

PERIOD DOUBLING PATTERNS AND EQUICONTINUITY OF ITERATES
IN ONE-DIMENSIONAL DYNAMICS

By

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We study some conditions under which continuous maps of the interval and the circle display chaotic or nonchaotic behaviour. In particular for interval maps, we introduce the notion of a period doubling pattern centered at a periodic orbit:

Let $f \in C^0(I, I)$. Suppose that f has a periodic orbit P of period $n \geq 1$. Let $\mathcal{G}$ be a countable collection of points consisting of one periodic orbit P_k of period $2^k n$ for each $k \geq 0$, satisfying the following three conditions:

(i) $P_0 = P$.

(ii) If $p_1 < p_2 < \dots < p_{2^{k+1}n}$ is the periodic orbit P_{k+1} of period $2^{k+1}n$, then between p_i and p_{i+1} , where $1 \leq i < 2^{k+1}n$ and i is odd, there exists a unique point z_i of P_k and no other point of period less than or equal to $2^k n$ of $\mathcal{G}$.

(iii) If p_i and p_{i+1} , where i is odd, are any two adjacent points of P_{k+1} and z_i is the unique point of P_k in $[p_i, p_{i+1}]$, then there exists j odd such that $\{f(p_i), f(p_{i+1})\} =$

$\{p_j, p_{j+1}\}$ and $f(z_i) \in (p_j, p_{j+1})$.

Then we say that f exhibits a period doubling pattern centered at P .

We prove that if a continuous map f of the interval is chaotic, in the sense that there exists a periodic orbit whose period is not a power of two, then f exhibits a period doubling pattern centered at a fixed point of f . We also prove that f exhibits a period doubling pattern centered at a fixed point of f if and only if the set of periodic points is not closed (i.e., f is not strongly nonchaotic).

Furthermore, we prove that if f has a periodic orbit of period $n > 1$ which is not a power of two, and $n \prec m$ in Sarkovskii's ordering, then f exhibits a period doubling pattern centered at a periodic orbit of period m . We also provide explicit ways for constructing such period doubling patterns.

For continuous maps of the circle, we observe that equicontinuity of the iterates of the map can restrict significantly the dynamic behaviour. If f is a continuous map on the circle, we let F_n denote the set of fixed points of f^n , for all positive n .

Then, we prove that $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if one of the following holds :

(i) f is conjugate to a rotation.

(ii) F_1 consists of exactly two distinct points and every other point on S^1 has period two.

(iii) F_1 consists of a single point and $F_2 = \bigcap_{n=1}^\infty f^n(S^1)$.

(iv) $F_1 = \bigcap_{n=1}^\infty f^n(S^1)$.

CHAPTER 1

INTRODUCTION

The study of topological dynamical systems has received a great deal of attention since the work of Poincaré and Birkhoff. As W. Gottschalk and G. Hedlund [22] (p.iii) mention: “Topological dynamics owes its origin to the classic work of Henri Poincaré and G.D.Birkhoff. It was Poincaré who first formulated and solved problems of dynamics as problems in topology. Birkhoff contributed fundamental concepts to topological dynamics and was the first to undertake its systematic development”.

We study several aspects of continuous maps on compact metric spaces, in particular the interval and the circle, which serve as the simplest examples of models for dynamical systems. However, the easy formulation combined with the complex behaviour of such systems and their close connection to the natural sciences makes their study very interesting. Let I denote a nondegenerate, compact, connected subset of the real line R , and S^1 denote the unit circle in the plane i.e. $S^1 = \{(x, y) \in R \times R : x^2 + y^2 = 1\}$.

If X is any compact metric space, we denote by $C^0(X, X)$ the set of continuous maps from X to itself. Let $f \in C^0(X, X)$. For any nonnegative integer n define f^n inductively by $f^n = f \circ f^{n-1}$, where f^0 is the identity map on X . A point $x \in X$ is a periodic point of f if there is a positive integer n such that $f^n(x) = x$.

The least such n is called the period of x . A point of period one is called a fixed point. Let F_n denote the fixed point set of f^n , for all $n \geq 1$ and $P(f) = P$ the set of periodic points of f .

The following striking theorem due to Sarkovskii [29], which appeared in 1964, became the basic tool for the study of the behaviour of continuous maps on the

interval:

Let N denote the set of positive integers. Let the positive integers be totally ordered in the following way:

$$3 \prec 5 \prec 7 \prec \dots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec \dots \prec 2^2 \cdot 3 \prec 2^2 \cdot 5 \prec \dots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

Theorem 1.0.1 *Let $f \in C^0(I, I)$. If f has a periodic point of period n and if $n \prec m$, then f also has a periodic point of period m .*

Sarkovskii's original proof of the above theorem is quite complicated, even in the relatively improved English version by Stefan [30]. Simpler proofs based on directed graphs were given by Block, Guckenheimer, Misiurewicz and Young [14] and Ho and Morris [23] independently.

For continuous maps of the circle the situation is quite different. Block [8] showed the following theorem:

Theorem 1.0.2 *Let $f \in C^0(S^1, S^1)$. Suppose that f has a fixed point and a point of period $n > 1$. Then at least one of the following holds:*

- (i) *for every integer m with $n < m$, f has a periodic point of period m ,*
- (ii) *for every integer m with $n \prec m$, f has a periodic point of period m .*

If $x \in X$, is a periodic point of period n for f , then the orbit of x consists of the n distinct points $x, f(x), \dots, f^{n-1}(x)$ and the trajectory of x , which we denote by $\gamma(x, f)$ or $\gamma(x)$ is the sequence consisting of the same points repeated periodically, i.e. $\gamma(x, f) = \gamma(x) = \{f^n(x)\}_{n \geq 0}$.

We define the limit set of a point $x \in X$ to be the set $\omega(x, f) = \omega(x) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} f^n(x)}$. Equivalently, $y \in \omega(x)$ if and only if y is a subsequential limit of the trajectory $\gamma(x)$, i.e. $f^{n_k}(x) \rightarrow y$ for some sequence of integers $n_k \rightarrow \infty$. We refer to [2], [5], [11] for more about the structure of ω -limit sets on the interval and the circle.

A point $x \in X$ is said to be *recurrent* if $x \in \omega(x)$, i.e., for any open set U containing x , there exists $n > 0$ such that $f^n(x) \in U$. Clearly, all periodic points are recurrent.

A point $x \in X$ is said to be *nonwandering* if for any open set U containing x , there exists $y \in U$ and $n > 0$ such that $f^n(y) \in U$.

Equivalently, x is recurrent if $f^{n_k}(x) \rightarrow x$ for some sequence of integers $n_k \rightarrow \infty$ and nonwandering if $f^{n_k}(x_k) \rightarrow x$ for some sequence of points $x_k \rightarrow x$ and some sequence of integers $n_k \rightarrow \infty$. Let R and Ω denote respectively the set of recurrent and nonwandering points.

We define *period doubling patterns* as follows:

Let $f \in C^0(I, I)$. Suppose that f has a periodic orbit P of period $n \geq 1$. Let $\mathcal{G}$ be a countable collection of points consisting of one periodic orbit P_k of period $2^k n$ for each $k \geq 0$, satisfying the following three conditions:

- (i) $P_0 = P$.
- (ii) If $p_1 < p_2 < \dots < p_{2^{k+1}n}$ is the periodic orbit P_{k+1} of period $2^{k+1}n$, then between p_i and p_{i+1} , where $1 \leq i < 2^{k+1}n$ and i is odd, there exists a unique point z_i of P_k and no other point of period less than or equal to $2^k n$ of $\mathcal{G}$.
- (iii) If p_i and p_{i+1} , where i is odd, are any two adjacent points of P_{k+1} and z_i is the unique point of P_k in $[p_i, p_{i+1}]$, then there exists j odd such that $\{f(p_i), f(p_{i+1})\} = \{p_j, p_{j+1}\}$ and $f(z_i) \in (p_j, p_{j+1})$.

Then we say that f exhibits a *period doubling pattern* centered at P .

Note that if p_i and p_{i+1} are as in condition (iii), then $f^{2^k n}(p_i) = p_{i+1}$ and $f^{2^k n}(p_{i+1}) = p_i$.

In the next two chapters we will be studying the existence of period doubling patterns, in relation to the underlying map f of the interval. We will prove, that if a map f is chaotic, then it exhibits a period doubling pattern centered at a fixed point of f .

There have been appeared several definitions of chaos in the last 20 years. For our purpose, we will follow the definition of Block and Coppel [10].

A map $f \in C^0(I, I)$ is said to be *chaotic*, if any of the following conditions are satisfied:

- (i) f has a periodic point whose period is not a power of 2
- (ii) f^n is turbulent for some positive integer n
- (iii) f has positive topological entropy.

For a definition of *turbulence*, mentioned in (ii), we refer to Section 2.5.

Topological entropy, which is mentioned in (iii), was first introduced by Adler, Konheim and McAndrew in 1965 [1], and it is a measure of the dynamical complexity of the map. See also [14] for more on the topological entropy of maps of the interval and the circle. We mainly use condition (i), in most of our results, as being the equivalent for a map to be chaotic.

We distinguish some *nonchaotic* maps as *uniformly nonchaotic* and *strongly nonchaotic*.

A continuous map $f : I \rightarrow I$ is uniformly nonchaotic, if every point $x \in I$ is approximately periodic, i.e. for every $\epsilon > 0$ there exists a periodic point y and a positive integer N such that, $d(f^n(x), f^n(y)) < \epsilon$, for all $n > N$.

A continuous map $f : I \rightarrow I$ is strongly nonchaotic, if every point $x \in I$ is asymptotically periodic, i.e. there exists a periodic point y such that, $d(f^n(x), f^n(y)) \rightarrow 0$, as $n \rightarrow \infty$.

Block and Coppel [10], have shown that the following are equivalent:

- (i) f is strongly nonchaotic
- (ii) $\omega(x, f)$ is finite for all $x \in I$.
- (iii) $P(f) = R(f)$
- (iv) $P(f)$ is a closed set.

In Section 2.4, we show that if $f \in C^0(I, I)$ has a periodic point whose period is not a power of 2, i.e. f is chaotic, then it exhibits a period doubling pattern centered at a fixed point of f . In particular, we provide a systematic way of producing a period doubling pattern centered at a fixed point of f .

We also show in Section 2.5, that f exhibits a period doubling pattern centered at a fixed point if and only if $P(f)$ is not closed. Therefore, period doubling patterns cannot be exhibited by strongly non-chaotic maps.

In Chapter 3, we prove that a chaotic map having a periodic orbit P of period n , exhibits a period doubling pattern centered at a periodic orbit Q of period m , whenever $n \prec m$ in Sarkovskii's ordering. We mainly adapt the terminology and notation of Misiurewicz and Nitecki [25].

We generalize the action of a map f on a periodic orbit, to the action of f on any f -invariant finite set P . In particular, we are interested in studying the case where P is a union of periodic orbits.

We define a pattern θ to be a map from $\{1, \dots, n\}$ to $\{1, \dots, n\}$. For a continuous map f and an f -invariant finite set P of cardinality n , we codify the action of f on P , by $f(p_k) = p_m$ if and only if $\theta(k) = m$, where p_k and p_m are points of P .

We say that the map f exhibits the pattern θ on P , and call P a representative of θ in f . We also say that a pattern θ forces another pattern η if every continuous map $f : I \rightarrow I$ which exhibits θ also exhibits η . Moreover, if P is a representative of θ in f , and Q is a representative of η in f , we say that P forces Q if and only if θ forces η . Finally, we say that P and Q have the same type if $\theta = \eta$.

It follows that if $F(P)$ denotes the set of all $f \in C^0(I, I)$ with a periodic orbit of the same type as P , then $F(P) \subseteq F(Q)$ if and only if P forces Q . For more about forcing relations we refer to [6] and [7]. Relationships between the permutations associated to periodic orbits of maps are examined in [15].

A useful pattern, for exhibiting period doubling patterns centered at P , will be what we call a *k-horseshoe*. In Section 3.1, we give the definition of a *k-horseshoe*, and we notice that 2-horseshoes force periodic orbits of period 3. We also define *extensions* and *reductions* of patterns, which turn out to be useful tools for producing horseshoes.

In the case of a periodic orbit P , the pattern θ exhibited on P is a cycle. We will define here, the extensions and reductions of cycles. Let $P = \{x_1 < \dots < x_{mk}\}$ be a periodic orbit of period $n = mk$, exhibiting the cycle θ , and Q be a periodic orbit of period k , exhibiting the cycle π . Then, we say that θ is a *k-extension* of π or π is a *k-reduction* of θ , if the m blocks $\{x_{k(j-1)+1} < \dots < x_{kj}\}$ ($j = 1, \dots, m$) of k consecutive points in P are permuted by θ , the permutation having the same type as Q . We refer to a 2-extension of a cycle as a doubling of it.

In Section 3.2, using results of [25], we show that if θ and π are cycles such that

- (1) θ forces π
- (2) θ is not an extension of π
- (3) π is not a doubling

then θ forces a 2-horseshoe extension of π .

Next, we start with a continuous map f which exhibits a cycle θ , and we assume that there exists a cycle π such that conditions (1), (2) and (3) of the previous statement are satisfied. We prove that f exhibits a period doubling pattern centered at the points of some representative of π in f .

This enables us to show that if $f \in C^0(I, I)$ has a periodic orbit of period $n > 1$ which is not a power of 2 and $n \prec m$ in Sarkovskii's ordering then there exists a period doubling pattern centered at a periodic orbit of period m .

The dynamics of continuous maps on the interval are not quite the same with that of the circle, as we have already seen with Theorem 1.0.1 and Theorem 1.0.2 .

Coven and Hedlund [20] showed that for maps on the interval $\overline{P} = \overline{R}$.

Nitecki [27] also showed that $\overline{P} = \Omega(f|\Omega)$ for piecewise monotone maps. Block and Coppel [10] give a proof of $\overline{P} = \overline{R} = \Omega(f|\Omega)$ for all continuous maps of the interval.

The first equality does not hold for maps of the circle, in this generality. Coven and Mulvey [21] showed that if $f \in C^0(S^1, S^1)$ has a periodic point, then $\overline{P} = \overline{R}$. The same result was obtained by Bae and Young [4] independently. Coven and Mulvey [21] also showed that for any $f \in C^0(S^1, S^1)$, $\overline{R} = \Omega(f|\Omega)$. For circle maps, it is shown [13] that if the set of periodic points is closed and nonempty, then every nonwandering point is periodic. The same result holds in the interval, without the requirement that $P(f)$ is nonempty.

In Chapter 4, we study continuous maps of the circle, with the property that the family of their iterates is equicontinuous. If $f \in C^0(S^1, S^1)$, let $\mathcal{F} = \{f, f^2, f^3, \dots\} = \{f^n\}_{n=1}^\infty$. Then the family of functions $\mathcal{F}$ is said to be equicontinuous, if given $\epsilon > 0$ there exists a $\delta > 0$, such that for all $x, y \in S^1$, $d(f^i(x), f^i(y)) < \epsilon$, for all $i \geq 1$, whenever $d(x, y) < \delta$.

Bruckner and Hu [17] showed that if $f \in C^0(I, I)$, then $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if $\bigcap_{n=1}^\infty f^n(I) = F_2$. This implies that $f \in C^0(I, I)$ can have periodic points of period at most 2. Therefore, the set of periodic points is closed, and f is strongly nonchaotic.

Our results are not quite the same for maps of the circle. We prove that if $f \in C^0(S^1, S^1)$, then $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if one of the following holds:

(i) f is conjugate to a rotation.

(ii) F_1 consists of exactly two distinct points and every other point on S^1 has period two.

(iii) F_1 consists of a single point and $F_2 = \bigcap_{n=1}^\infty f^n(S^1)$.

(iv) $F_1 = \bigcap_{n=1}^\infty f^n(S^1)$.

We will show that (i) corresponds to the cases where either the set of periodic points of f is empty and f is conjugate to an irrational rotation, or every point is periodic with the same period, in which case f is conjugate to a rational rotation.

In the remaining cases (ii), (iii) and (iv), f cannot have periodic points of period greater than 2.

CHAPTER 2

PERIOD DOUBLING PATTERNS CENTERED AT FIXED POINTS

In this chapter, we will see how chaotic maps on the interval, exhibit period doubling patterns centered at a fixed point. Hence we assume that $f \in C^0(I, I)$ has a periodic orbit, whose period is not a power of 2.

The main results of this chapter can be found in Theorem 2.4.2 and Theorem 2.5.4. In Theorem 2.4.2, we prove that if $f \in C^0(I, I)$ has a periodic point whose period is not a power of 2, then it exhibits a period doubling pattern centered at a fixed point of f . In Theorem 2.5.4, we prove that $f \in C^0(I, I)$ exhibits a period doubling pattern centered at a fixed point if and only if the set of periodic points is not closed.

In Section 2.1, we introduce the notion of Markov graphs, and we show how these graphs guarantee the existence of periodic orbits of certain periods. We also define Stefan, simple and strongly simple periodic orbits and we recall some well-known results regarding such orbits.

In Section 2.2, we prove that if $f \in C^0(I, I)$ has a periodic orbit of period 3, then it exhibits a period doubling pattern centered at a fixed point of f .

In Section 2.3, we first show that if the linearization of a cyclic permutation exhibits a period doubling pattern centered at a fixed point, then any continuous map which exhibits the cyclic permutation also exhibits a period doubling pattern centered at a fixed point.

Using this result, we prove that if $f \in C^0(I, I)$ has a periodic orbit of period 6, then it exhibits a period doubling pattern centered at a fixed point.

In Section 2.4, using results from Section 2.3, we prove the more general lemma that, if $f \in C^0(I, I)$ has a periodic orbit of period $2^k \cdot 3$, then f exhibits a period doubling pattern centered at a fixed point. This enables us to show the first main result in Theorem 2.4.2.

In Section 2.5, we use turbulence and results from [10], to show the second main result in Theorem 2.5.4.

2.1 Definitions and Preliminaries

We start with several lemmas and techniques that guarantee the existence of periodic orbits of certain periods. Let $f \in C^0(I, I)$.

The following lemma is a consequence of the Intermediate Value Theorem, and guarantees the existence of a fixed point of f .

Lemma 2.1.1 *If J is a compact subinterval such that $J \subseteq f(J)$, then f has a fixed point in J .*

The above lemma can be generalized, as follows. For a proof see [10](p.7):

Lemma 2.1.2 *If $J_0, J_1, \dots, J_m$ are compact subintervals such that $J_k \subseteq f(J_{k-1})$ ($1 \leq k \leq m$), then there is a compact subinterval $L \subseteq J_0$ such that $f^m(L) = J_m$ and $f^k(L) \subseteq J_k$ ($1 \leq k < m$).*

If also $J_0 \subseteq J_m$, then there exists a point y such that $f^m(y) = y$ and $f^k(y) \in J_k$ ($1 \leq k < m$).

We denote by $\langle a, b \rangle$ a closed subinterval of I with endpoints a and b .

Suppose that f has a finite invariant set P of cardinality $n > 1$. Let $x_1 < \dots < x_n$ be the distinct points of P and set $I_j = [x_j, x_{j+1}]$ ($1 \leq j < n$). With this finite invariant set we associate a *directed graph*, or *digraph*, in the following way. The vertices of the directed graph are the subintervals $I_1, \dots, I_{n-1}$, and there is an arc $I_j \rightarrow I_k$ if I_k is contained in the closed interval $\langle f(x_j), f(x_{j+1}) \rangle$ with endpoints $f(x_j)$ and $f(x_{j+1})$.

In this case we also say that I_j covers I_k . We often call such a directed graph a *Markov graph*, because of its relation to adjacency matrices and topological Markov chains.

A loop $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{n-1} \rightarrow J_0$ of length n in the digraph will be said to be *nonrepetitive* if it does not consist entirely of a loop of smaller length described several times.

The following lemma follows from Lemma 2.1.2.

Lemma 2.1.3 *Let P be a finite set of points such that $f(P) \subseteq P$. If the corresponding Markov graph contains a loop $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_0$ of length m , then there is a fixed point y of f^m such that $f^k(y) \in J_k$, for $0 \leq k < m$.*

A consequence of the above lemma is that if the loop $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{m-1} \rightarrow J_0$ is nonrepetitive, and the points $f^k(y)$ lie in the interior of J_k , for $0 \leq k < m$, then f has a periodic orbit of period m .

Definition 2.1.4 *Suppose that f has a periodic orbit P of odd period $n > 1$. Let c be the midpoint of this orbit. Then we say that P is a Stefan orbit if the points have either of the following two configurations:*

$$(i) f^{n-1}(c) < f^{n-3}(c) < \dots < f^2(c) < c < f(c) < \dots < f^{n-2}(c)$$

or

$$(ii) f^{n-2}(c) < \dots < f(c) < c < f^2(c) < \dots < f^{n-3}(c) < f^{n-1}(c).$$

Cosnard [19] showed the following interesting result:

Proposition 2.1.5 *If f has a periodic orbit of odd period $n > 1$, then f has a Stefan orbit of period n .*

It is sometimes convenient to think of a periodic orbit as an ordered pair (P, π) , where P is a finite subset of the interval and π a cyclic permutation of P . We also refer to P as the periodic orbit, with π understood. In this way, we talk about

periodic orbits without involving any map f of the interval. We say a periodic orbit P is an orbit of such a map f , if $f(x) = \pi(x)$ for every $x \in P$. In general, P can be a union of periodic orbits, and π a multicyclic permutation, as shown below:

Let $f \in C^0(I, I)$ and $P = \{p_1 < p_2 < \dots < p_n\}$ be a finite f -invariant set. The action of f on P can be codified in the map $\theta: \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ defined by $f(p_i) = p_{\theta(i)}$, $i = 1, \dots, n$. To stress the combinatorial role of θ we refer to any map of $\{1, \dots, n\}$ to itself as a *pattern* on n elements. The number n is the degree of θ .

We say that the map f exhibits the pattern θ on P and call P a *representative* of θ in f .

When P is a single periodic orbit, the pattern θ represented by P is a *cycle* (cyclic permutation) and we say that f has a periodic orbit of type θ .

More generally, a bijective pattern θ is a (possibly multicyclic) permutation, and a representative of θ is a finite union of periodic orbits.

Definition 2.1.6 *Let (P, π) be a periodic orbit of period n . We define a simple periodic orbit, by induction on the highest power of 2 which divides the period.*

When n is odd, we say that P is simple if $n = 1$ or if $n > 1$ and P is of Stefan type.

When $n = qm$, where q is odd and $m = 2^s > 1$, we say that P is simple if the left and right halves of P each form simple orbits of π^2 with period $n/2$.

It follows from this definition that, if $q > 1$ and $m > 1$, then $P = \{x_1 < \dots < x_n\}$ is a simple orbit if and only if the m blocks $\{x_{q(k-1)+1}, \dots, x_{qk}\}$ ($k = 1, \dots, m$) of q consecutive points each form a simple orbit of π^m with period q , and the blocks themselves are permuted by π like a simple orbit with period m .

Definition 2.1.7 *Let (P, π) be a periodic orbit of period $n = qm$, where q is odd and $m = 2^s$. We say P is strongly simple if it is simple and either n is a power of 2 ($q = 1$), or n is odd ($m = 1$), or $q > 1$, $m > 1$, and π maps each of the m blocks of q consecutive points monotonically onto another such block, with one exception.*

Block and Coppel showed in [9] the following important theorem:

Theorem 2.1.8 *Let $f \in C^0(I, I)$. If f has an orbit of period n , then f has a strongly simple orbit of period n .*

Definition 2.1.9 *Let $P = \{p_1 < p_2 < \dots < p_n\}$ be a finite invariant set, consisting of a union of periodic orbits, and let π be a multicyclic permutation represented by P . There is a unique map $f : [p_1, p_n] \rightarrow [p_1, p_n]$ satisfying*

- (i) $f(p_i) = \pi(p_i)$ for $i = 1, \dots, n$,
- (ii) $f|_{[p_i, p_{i+1}]}$ is linear for $i = 1, \dots, n-1$.

This map f is called the linearization of P .

Definition 2.1.10 *Let $P = \{p_1 < p_2 < \dots < p_n\}$ and $Q = \{q_1 < q_2 < \dots < q_m\}$, be finite invariant sets, and let π and η be patterns represented by P and Q , respectively. We say P and Q have the same type if $n = m$ and $\pi(p_i) = p_j$ if and only if $\eta(q_i) = q_j$.*

Remark: suppose that $f \in C^0(I, I)$ exhibits the cyclic permutation π on P , and that $g \in C^0(I, I)$ exhibits the cyclic permutation η on Q . Then, if P and Q have the same type, we also say that $f|_P$ and $g|_Q$ are of the same type too.

Definition 2.1.11 *A periodic orbit (P, π) of period n , with $P = \{p_1 < p_2 < \dots < p_n\}$, will be said to be a 2-extension (doubling) of a periodic orbit (Q, η) of period m if $n = 2m$ and if the m blocks $\{p_{2j-1} < p_{2j}\}$ ($j = 1, \dots, m$) of 2 consecutive terms in P are permuted by π , the permutation having the same type as Q .*

We have already defined in Chapter 1 what a period doubling pattern centered at a periodic orbit P is. In this chapter, we are primarily interested about period doubling patterns centered at a fixed point. Consequently, we recall the definition adapted in this case:

Definition 2.1.12 Let $f \in C^0(I, I)$. Suppose that f has a fixed point p in I . Let $\mathcal{G}$ be a countable collection of points consisting of one periodic orbit P_k of period 2^k for each $k \geq 0$, satisfying the following three conditions:

(i) $P_0 = \{p\}$.

(ii) If $p_1 < p_2 < \dots < p_{2^{k+1}}$ is the periodic orbit P_{k+1} of period 2^{k+1} , then between p_i and p_{i+1} , where $1 \leq i < 2^{k+1}$ and i is odd, there exists a unique point z_i of P_k and no other point of period less than or equal to 2^k of $\mathcal{G}$.

(iii) If p_i and p_{i+1} , where i is odd, are any two adjacent points of P_{k+1} and z_i is the unique point of P_k in $[p_i, p_{i+1}]$, then there exists j odd such that $\{f(p_i), f(p_{i+1})\} = \{p_j, p_{j+1}\}$ and $f(z_i) \in (p_j, p_{j+1})$.

Then we say that f exhibits a period doubling pattern centered at p .

2.2 Existence of an Orbit of Period 3

Having in mind Sarkovskii's ordering, we first assume that f has a periodic orbit of period 3, and we prove that a period doubling pattern centered at a fixed point, exists in this case.

Lemma 2.2.1 If $f \in C^0(I, I)$ has a periodic orbit of period 3, then it exhibits a period doubling pattern centered at a fixed point of f .

Proof: Let c be the midpoint of the periodic orbit of period 3. Then the points of this orbit have the order $f^2(c) < c < f(c)$ or the reverse order $f(c) < c < f^2(c)$. Without loss of generality assume $f^2(c) < c < f(c)$.

If $K = [a, b]$ and $L = [c, d]$, then we say that K covers L in an orientation preserving way, if $f(a) \leq c$ and $f(b) \geq d$, and we denote it by $K \xrightarrow{P} L$. We also say that K covers L in an orientation reversing way, if $f(a) \geq d$ and $f(b) \leq c$, and we denote it by $K \xrightarrow{R} L$.

Note that $[c, f(c)] \rightarrow [c, f(c)]$, i.e. $[c, f(c)]$ covers itself, and therefore there is a loop of length one. Consequently there is a fixed point of f in $[c, f(c)]$, which we denote by 1 as shown in Figure 1:



Fig. 1

Now let $A = [f^2(c), c]$ and $B = [1, f(c)]$. Then the loop $A \xrightarrow{P} B \xrightarrow{R} A$ has length two. Moreover, the intervals A and B do not have any endpoint in common since the interval $(c, 1)$ lies between them. Therefore there is a periodic orbit of period two, consisting of $2_1 \in A$ and $2_2 \in B$.

We have constructed the period doubling pattern up to periodic orbits of period two, as shown in Figure 2:



Fig. 2

Now let $A_1 = [f^2(c), 2_1]$, $A_2 = [c, 1]$, $B_1 = [1, 2_2]$ and $B_2 = [2_2, f(c)]$. Then the loop $A_1 \xrightarrow{P} B_1 \xrightarrow{R} A_2 \xrightarrow{R} B_2 \xrightarrow{R} A_1$ has length four. Moreover, the intervals A_1 and A_2 do not have any endpoint in common, since the interval $(2_1, c)$ lies between them. Therefore there is a periodic orbit of period four, consisting of $4_1 \in A_1$, $4_2 \in B_1$, $4_3 \in A_2$ and $4_4 \in B_2$.

Note that each A_i or B_i for $i = 1, 2$ is a subset of A or B , respectively, except A_2 which is now to the right of c . This forces A_2 to map onto B_2 in an orientation reversing way, as shown in Figure 3:

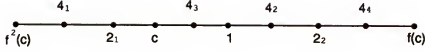


Fig. 3

Now let $A_{11} = [f^2(c), 4_1]$, $A_{12} = [4_1, 2_1]$, $A_{21} = [2_1, c]$, $A_{22} = [4_3, 1]$,

$B_{11} = [1, 4_2]$, $B_{12} = [4_2, 2_2]$, $B_{21} = [2_2, 4_4]$, $B_{22} = [4_4, f(c)]$.

Then the loop

$$A_{11} \xrightarrow{P} B_{11} \xrightarrow{R} A_{22} \xrightarrow{R} B_{21} \xrightarrow{R} A_{12} \xrightarrow{P} B_{12} \xrightarrow{R} A_{21} \xrightarrow{P} B_{22} \xrightarrow{R} A_{11}$$

has length eight. Notice that for the construction of this loop, we follow the orbit of period 4, except that the second time we meet 4_3 , we switch to the other side of c . Moreover, the intervals A_{21} and A_{22} do not have any endpoint in common, since the interval $(c, 4_3)$ lies between them. Therefore there is a periodic orbit of period eight, consisting of $8_1 \in A_{11}$, $8_2 \in B_{11}$, $8_3 \in A_{22}$, $8_4 \in B_{21}$, $8_5 \in A_{12}$, $8_6 \in B_{12}$, $8_7 \in A_{21}$, $8_8 \in B_{22}$.

Again note that each A_{ij} or B_{ij} is a subset of A_i or B_i for $i, j = 1, 2$, except A_{21} which is now to the left of c .

Also, 8_i and 8_{i+4} , where $i = 1, 2, 3, 4$ are adjacent points, and there is a unique point of period four between them constructed in the previous stage and $\{f(8_i), f(8_{i+4})\} = \{8_{i+1}, 8_{(i+5)(\text{mod } 8)}\}$ ($i = 1, 2, 3, 4$).

This forces A_{21} to map onto B_{22} in an orientation preserving way and thus the first four arrows of the loop are characterized by PRRR like the orbit of period four, whereas the other four are characterized by PRPR.

At this stage, we have constructed the period doubling pattern for orbits of period one, two, four and eight, as shown in Figure 4:

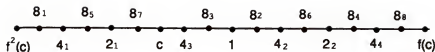


Fig. 4

At the next stage let

$$A_{111} = [f^2(c), 8_1], A_{112} = [8_1, 4_1], A_{121} = [4_1, 8_5], A_{122} = [8_5, 2_1],$$

$$A_{211} = [2_1, 8_7], A_{212} = [c, 4_3], A_{221} = [4_3, 8_3], A_{222} = [8_3, 1],$$

$$B_{111} = [1, 8_2], B_{112} = [8_2, 4_2], B_{121} = [4_2, 8_6], B_{122} = [8_6, 2_2],$$

$$B_{211} = [2_2, 8_4], B_{212} = [8_4, 4_4], B_{221} = [4_4, 8_8], B_{222} = [8_8, f(c)].$$

Then the loop

$$\begin{aligned} A_{111} &\xrightarrow{P} B_{111} \xrightarrow{R} A_{222} \xrightarrow{R} B_{211} \xrightarrow{R} A_{122} \xrightarrow{P} B_{122} \xrightarrow{R} A_{211} \xrightarrow{P} B_{221} \xrightarrow{R} \\ A_{112} &\xrightarrow{P} B_{112} \xrightarrow{R} A_{221} \xrightarrow{R} B_{212} \xrightarrow{R} A_{121} \xrightarrow{P} B_{121} \xrightarrow{R} A_{212} \xrightarrow{R} B_{222} \xrightarrow{R} A_{111} \end{aligned}$$

has length sixteen. Notice that for the construction of this loop, we follow the orbit of period 8, except that the second time we meet 8_7 we switch to the other side of c . Moreover, the intervals A_{211} and A_{212} do not have any endpoint in common since the interval $(8_7, c)$ lies between them. Therefore there is a periodic orbit of period sixteen. We call the points of this orbit $16_1, \dots, 16_{16}$ and note again that each A_{ijk} or B_{ijk} with $i, j, k = 1, 2$ is a subset of A_{ij} or B_{ij} respectively, except A_{212} which is now to the right of c .

This forces A_{212} to map onto B_{222} in an orientation reversing way, and thus the first eight arrows of the loop are characterized by PRRRPRPR like the periodic orbit of period eight, whereas the other eight are characterized by PRRRPRRR.

At this stage, we have constructed the period doubling pattern for orbits of period one, two, four, eight and sixteen, as shown in Figure 5:

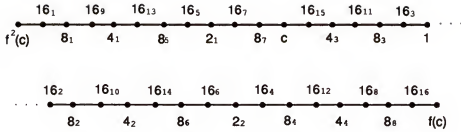


Fig. 5

Now, we use induction on the period of the periodic orbits of f . We assume that we have constructed the period doubling pattern for orbits of period up to 2^n , by defining the appropriate A 's and B 's and by constructing the corresponding loops. Recall that the A 's lie to the left of the fixed point 1, and the B 's to the right of 1.

We want to show that there is an orbit of period 2^{n+1} , sitting in the appropriate intervals according to the definition of the period doubling pattern.

We use the following notation: let i^k denote $\underbrace{i \dots i}_{k\text{-times}}$, for $i = 1, 2$, where 1^k or 2^k appears as a subscript to the intervals A 's and B 's.

The periodic orbit of period 2^n is obtained by a loop of A 's and B 's, where each A or B is characterized by an index with $n - 1$ symbols. Recall that we have labeled the points of this orbit, according to where they map onto, with respect to the corresponding loop. If $(2^n)_1, (2^n)_2, \dots, (2^n)_{2^n}$ denote the points of the periodic orbit of period 2^n , then the loop that guarantees a periodic orbit of period 2^n has the following form:

$$A_{1^{n-1}} \xrightarrow{P} B_{1^{n-1}} \xrightarrow{R} A_{2^{n-1}} \xrightarrow{R} B_{21^{n-2}} \xrightarrow{R} A_{12^{n-2}} \xrightarrow{P} \dots \longrightarrow B_{2^{n-1}} \xrightarrow{R} A_{1^{n-1}}$$

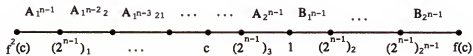


Fig. 6

To obtain a periodic orbit of period 2^{n+1} , we use intervals obtained by the periodic orbit of period 2^n in the following way: each A_q or B_q , where q is a string of 1's and 2's of length $n-1$, splits into two subintervals, labeled by indices of length n , by adding '1' to the left subinterval and '2' to the right one. These have the points of the periodic orbits of period 2^j , for $j = 0, 1, 2, \dots, n$ as endpoints, with one exception: the interval with c as its right or left endpoint, "splits" into two subintervals, such that the right or left subinterval switches to the other side of c respectively, for example if $[c, (2^{n-2})_{2^{n-2}-1}] = A_{\underbrace{2121\dots 212}_{n-1}}$ then it splits into $[(2^{n-1})_{2^{n-1}-1}, c]$ and $[(2^n)_{2^n-1}, (2^{n-2})_{2^{n-2}-1}]$.

The loop of length 2^{n+1} is obtained, by starting with $A_{1^n} = [f^2(c), (2^n)_1]$, and following the orbit of $(2^n)_1$. Each time, we choose the interval with endpoints $(2^n)_i$ ($i = 1, \dots, 2^n$) and the adjacent point to $(2^n)_i$ covered by the previous interval in the loop. The last interval $B_{2^{n-1}} = [(2^{n-1})_{2^{n-1}}, (2^n)_{2^n}]$ covers $A_{1^{n-2}} = [(2^n)_1, (2^{n-1})_1]$. Again, we follow the orbit of $(2^n)_1$, with the exception that, when we meet the interval having c as an endpoint, we switch to the other side of c .

The loop of length 2^{n+1} respects the form of the loop of length 2^n of the previous stage concerning the P-R notation except that it is repeated twice and the next to the last arrow will switch from P to R or conversely, since the interval $A_{2121\dots 212}$ or $A_{2121\dots 21}$ switches to the other side of c .

Thus the loop that guarantees a periodic orbit of period 2^{n+1} has the form

$$A_{1^n} \xrightarrow{P} B_{1^n} \xrightarrow{R} A_{2^n} \xrightarrow{R} B_{2^{n-1}} \xrightarrow{R} A_{1^{n-1}} \xrightarrow{P} \dots \longrightarrow B_{2^{n-1}} \xrightarrow{R} A_{1^{n-2}} \longrightarrow \dots \longrightarrow B_{2^n} \xrightarrow{R} A_{1^n}$$

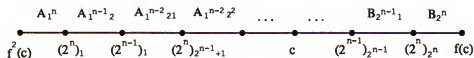


Fig. 7

This step completes the induction and the proof of the lemma. ■

2.3 Linearization and Existence of an Orbit of Period 6

Having proved that a period doubling pattern centered at a fixed point exists in the case that f has a periodic orbit of period 3, we want to continue and prove the existence of such a period doubling pattern, when f has a periodic orbit of period $2 \cdot 3$.

A proof of the following lemma can be found in [10](p.170).

Lemma 2.3.1 *Let P be a periodic orbit and f its linearization. Let Q be a periodic orbit of f , with period m , and suppose there are adjacent points x and y of Q such that none of the intervals $f^i[x, y]$ ($0 \leq i < m$) contains a point of P in its interior.*

Then P is a doubling, and P and Q have the same type.

We will next see that from now on we can work with linearizations, since if the linearization of an orbit (P, π) has a period doubling pattern centered at a fixed point, then any continuous map exhibiting π on P has a period doubling pattern centered at a fixed point too.

Proposition 2.3.2 *Let $g \in C^0(I, I)$ exhibit a cyclic permutation π on a periodic orbit P whose period is not a power of two. Let f denote the linearization of π on P . If f exhibits a period doubling pattern centered at a fixed point, then g exhibits a period doubling pattern centered at a fixed point.*

Proof: Let $\{P_j\}_{j=0}^\infty$ denote the collection of periodic orbits of the period doubling pattern of f , where P_0 consists of a fixed point of f and P_j is a periodic orbit of period 2^j , for each $j \geq 0$.

We will construct a period doubling pattern $\{Q_j\}_{j=0}^\infty$ centered at a fixed point of g , using induction. First note that there exists an interval joining adjacent points of P , which covers itself under f , since the fixed point of the period doubling pattern

$\{P_j\}_{j=0}^\infty$ exists. The same interval covers itself under g , and therefore Q_0 exists, too. Assume that $\{Q_j\}_{j=0}^N$ has been constructed, so that $P \cup P_0 \cup P_1 \cup \dots \cup P_N$ and $P \cup Q_0 \cup Q_1 \cup \dots \cup Q_N$ have the same type. For each non-negative integer i , let $A_i = \{x : f^i(x) \in P \cup P_0 \cup P_1 \cup \dots \cup P_N\}$. Note that $A_i = f^{-1}(A_{i-1})$ and A_i is finite for each positive integer i .

We claim that each interval joining adjacent points of A_{2N+1} contains at most one point of P_{N+1} . To see this, first note that the orbits P and P_{N+1} are of different type, since the period of P is not a power of 2 and the period of P_{N+1} is 2^{N+1} . Now let $\tilde{A}_n = \{x : f^n(x) \in P\}$. It follows from Lemma 2.3.1 applied to P and P_{N+1} that each interval joining adjacent points of $\tilde{A}_{2N+1}$ contains at most one point of P_{N+1} . Since $\tilde{A}_{2N+1} \subseteq A_{2N+1}$, each interval joining adjacent points of A_{2N+1} is contained in some interval joining adjacent points of $\tilde{A}_{2N+1}$. Hence, we conclude that each interval joining adjacent points of A_{2N+1} contains at most one point of P_{N+1} .

Now let $B_0 = P \cup Q_0 \cup Q_1 \cup \dots \cup Q_N$, so that B_0 and A_0 are of the same type. We define $B_1, B_2, \dots, B_{2N+1}$ inductively as follows. Assuming B_{n-1} has been defined, we take B_n to be a subset of $g^{-1}(B_{n-1})$ such that:

- (i) $B_n \supseteq B_{n-1}$,
- (ii) B_n and A_n have the same cardinality,
- (iii) if $A_n = \{y_1 < \dots < y_n\}$ and $B_n = \{z_1 < \dots < z_n\}$, then $f|_{A_n}$ and $g|_{B_n}$

have the same type.

The finite sets A_{2N+1}, B_{2N+1} are invariant under f, g respectively, and the Markov graphs of (A_{2N+1}, f) and (B_{2N+1}, g) are the same. By following the orbit of P_{N+1} , we obtain a loop in the Markov graph of (A_{2N+1}, f) . This same loop in the Markov graph of (B_{2N+1}, g) yields a periodic orbit Q_{N+1} of g . Note that each interval joining adjacent points of B_{2N+1} contains at most one point of Q_{N+1} . Therefore $f|_{P \cup P_0 \cup P_1 \cup \dots \cup P_{N+1}}$ and $g|_{P \cup Q_0 \cup Q_1 \cup \dots \cup Q_{N+1}}$ have the same type, and $Q_0 \cup Q_1 \cup \dots \cup Q_{N+1}$

respects the definition of a period doubling pattern. The induction step is now complete. ■

We are now ready to show that a period doubling pattern centered at a fixed point exists, when f has an orbit of period 6. By Proposition 2.3.2, it suffices to prove this for the linearization.

Lemma 2.3.3 *If $f \in C^0(I, I)$ has a periodic orbit of period 6, then it exhibits a period doubling pattern centered at a fixed point of f .*

Proof: By Theorem 2.1.8, we can assume that f has a strongly simple orbit P of period 6 and show the lemma for such an orbit. This means that the left and right half of this orbit each form simple orbits of f^2 with period 3 and one of the two blocks of 3 consecutive points maps monotonically onto the other.

By Proposition 2.3.2, it suffices to prove the lemma for the linearization of P , which we call L . Let B_1 and B_2 denote the convex hulls of the two blocks of 3 consecutive points, with $B_1 < B_2$, meaning that all the points of B_1 lie to the left of all the points of B_2 .

By Lemma 2.2.1 applied to L^2 , in B_1 , we obtain a period doubling pattern centered at a fixed point of L^2 in the convex hull of B_1 , which we denote by 1. Since, $L(B_1) = B_2$, there exists a point $L(1) \in B_2$, such that $L^2(1) = 1$. Hence, we obtain a periodic orbit of period 2, of L .

Next, we denote by $2_1, 2_2$, the points of the orbit of period 2 of the period doubling pattern of L^2 in B_1 . Then $L(2_1)$ and $L(2_2)$ are points which lie, by linearity of L , in intervals of B_2 , respecting the definition of a period doubling pattern. Thus, we obtain a periodic orbit of period 4, of L .

Proceeding in this way, by looking at the images of the points of the periodic orbit of period 2^k of L^2 , we obtain points in B_2 , which by linearity lie in appropriate intervals of B_2 according to the definition of a period doubling pattern.

We have produced periodic orbits of period 2^k for $k > 1$, of L , respecting the period doubling pattern. To produce the fixed point of L , notice that the interval with left endpoint the largest point of B_1 , and right endpoint the smallest point of B_2 , covers itself. Hence there exists a fixed point of L , sitting at the center of the pattern, and thus L exhibits a period doubling pattern centered at a fixed point of L . Thus, f exhibits a period doubling pattern centered at a fixed point of f . ■

We are going to demonstrate the above result by taking a specific strongly simple periodic orbit of period 6 and constructing a period doubling pattern centered at a fixed point of f . By Proposition 2.3.2, we may assume that f is the linearization of the periodic orbit. Let $a_1 < b_1 < c_1$ and $a_2 < b_2 < c_2$ be the 2 blocks of 3 consecutive points. Without loss of generality assume that the block $\{a_1, b_1, c_1\}$ maps in a monotonically increasing way onto the block $\{a_2, b_2, c_2\}$ so that $f(a_1) = a_2$, $f(b_1) = b_2$ and $f(c_1) = c_2$. Then, either $f(a_2) = b_1$, $f(b_2) = c_1$ and $f(c_2) = a_1$ or $f(a_2) = c_1$, $f(b_2) = a_1$ and $f(c_2) = b_1$. Again, without loss of generality assume that the first case holds.

Thus $a_1 \xrightarrow{f} a_2 \xrightarrow{f} b_1 \xrightarrow{f} b_2 \xrightarrow{f} c_1 \xrightarrow{f} c_2 \xrightarrow{f} a_1$. First note that the orbit of period 2 is simple, so that the left half of the orbit, which is the interval $[a_1, c_1]$, maps under f onto the right half of the orbit, which is the interval $[a_2, c_2]$ and viceversa. In particular the interval $[c_1, a_2]$ covers itself and thus it contains a fixed point of f , which we denote by 1, as shown in Figure 8:

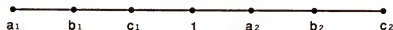


Fig. 8

In the next three figures, we will illustrate the period doubling pattern centered at the above fixed point, for orbits up to period 8:

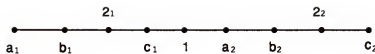


Fig. 9

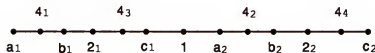


Fig.10

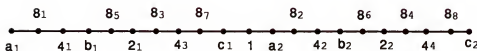


Fig. 11

Proceeding inductively, we can construct a period doubling pattern centered at a fixed point, as shown in Lemma 2.3.3.

2.4 Chaotic Maps Exhibit a Period Doubling Pattern

Lemma 2.4.1 *If $f \in C^0(I, I)$ has a periodic orbit of period $3 \cdot 2^k$, then it exhibits a period doubling pattern centered at a fixed point of f .*

Proof: By Theorem 2.1.8, we can assume that f has a strongly simple orbit P of period $3 \cdot 2^k$ and show the lemma for such an orbit. This means that the orbit is simple and each of the 2^k blocks of 3 consecutive points maps monotonically onto another such block, with one exception.

By Proposition 2.3.2, it suffices to prove the lemma for the linearization of P . Therefore, without loss of generality, let f denote the linearization of P .

Let $B_1, B_2, \dots, B_{2^k}$ denote the convex hulls of the 2^k blocks of 3 consecutive points, where $f(B_i) = B_{i+1}$, for $i = 1, \dots, 2^k - 1$, and $f(B_{2^k}) = B_1$.

Notice that $f^{2^k}|_{B_1}$, has a periodic orbit of period 3. By Lemma 2.2.1 applied to f^{2^k} in B_1 , we obtain a period doubling pattern centered at a fixed point of f^{2^k} in B_1 , which we call 1. Since, $f(B_i) = B_{i+1}$, for all $i = 1, \dots, 2^k - 1$, and $f(B_{2^k}) = B_1$, there exist points $f^i(1) \in B_{i+1}$, ($i = 1, \dots, 2^k - 1$), such that $f^{2^k}(1) = 1$. Hence, we obtain a periodic orbit of period 2^k of f , consisting of the points $1, f(1), f^2(1), \dots, f^{2^k-1}(1)$.

Next, we denote by $2_1, 2_2$, the points of the orbit of period 2 of the period doubling pattern of f^{2^k} in B_1 . Then $f^i(2_1), f^i(2_2) \in B_{i+1}$, ($i = 1, \dots, 2^k - 1$), are points which lie, by linearity of f , in appropriate intervals, respecting the definition of a period doubling pattern.

Therefore, $f^i(1) \in \langle f^i(2_1), f^i(2_2) \rangle$, ($i = 1, \dots, 2^k - 1$), and hence we obtain a periodic orbit of period $2 \cdot 2^k$ of f , consisting of the points $2_j, f(2_j), f^2(2_j), \dots, f^{2^k-1}(2_j)$, for $j = 1$ and 2 .

Proceeding in this way, by looking at the images of the points of the periodic orbit of period 2^n ($n \geq 0$) of f^{2^k} , we obtain points in the B_i 's, which by linearity, lie in appropriate intervals, according to the definition of a period doubling pattern.

Hence, we obtain periodic orbits of period 2^n , of f , for all $n \geq k$.

Let $S = \overline{\min\{x : x \in B_i\}_{i=1}^{2^k}, \max\{x : x \in B_i\}_{i=1}^{2^k}} \cup \bigcup_{i=1}^{2^k} B_i$, and $J_1, J_2, \dots, J_{2^k-1}$ denote the components of S , numbered so that $J_1 < J_2 < \dots < J_{2^k-1}$.

We will show that there exist periodic orbits of period 2^j for $j = 0, 1, 2, \dots, k-1$, lying in the J_j 's for $j = 1, 2, \dots, 2^k-1$ and respecting the definition of a period doubling pattern. (See Fig. 12 for an example).

First note that the orbit of period $3 \cdot 2^k$ is simple, so that the left half of the orbit, which we denote by L, maps under f onto the right half of the orbit, which we denote by R, and viceversa R maps under f onto L. It follows that the middle

interval determined by the right endpoint of L and the left endpoint of R, which is $J_{2^{k-1}}$, covers itself, and thus it contains a fixed point of f .

Now look at the left and right halves of L, which we denote by LL and LR respectively. Since L is a simple orbit itself under f^2 , it follows that every point of LL maps under f^2 into LR and viceversa. In particular the right endpoint of LL maps under f^2 into LR and the left endpoint of LR maps under f^2 into LL and hence the interval determined by these two endpoints, namely $J_{2^{k-2}}$, covers itself under f^2 .

Similarly, if RL and RR denote the left and right halves of R, respectively, then again by simplicity of the orbits, the right endpoint of RL maps under f^2 into RR and the left endpoint of RR maps under f^2 into RL. Hence the interval determined by these two endpoints, namely $J_{2^{k-2}+2^{k-1}}$, covers itself under f^2 .

Moreover, $J_{2^{k-2}} \longrightarrow J_{2^{k-2}+2^{k-1}} \longrightarrow J_{2^{k-2}}$. Thus, there exists a periodic orbit of period 2.

At the next stage, we concentrate at the left and right halves of LL,LR,RL and RR. The left halves are denoted by adding an L to the existing notation and the right halves by adding an R. Then it follows that the following intervals cover themselves under f^4 :

- (i) $J_{2^{k-3}}$ which lies in between LLL and LLR
- (ii) $J_{2^{k-3}+2^{k-2}}$ which lies in between LRL and LRR
- (iii) $J_{2^{k-3}+2^{k-1}}$ which lies in between RLL and RLR
- (iv) $J_{2^{k-3}+2^{k-2}+2^{k-1}}$ which lies in between RRL and RRR.

Moreover, $J_{2^{k-3}} \longrightarrow J_{2^{k-3}+2^{k-1}} \longrightarrow J_{2^{k-3}+2^{k-2}} \longrightarrow J_{2^{k-3}+2^{k-2}+2^{k-1}}$. Thus, there exists a periodic orbit of period 4, which together with the fixed point and the periodic orbit of period 2 satisfy the definition of the period doubling pattern up to this stage.

Proceeding in the same way, by simplicity of the orbits of the left and right halves at each stage, we obtain periodic orbits of period 2^j for all $j = 0, 1, 2, \dots, k-1$.

These are exactly $(1+2+2^2+\dots+2^{k-1}) = 2^k - 1$ periodic points and are located inside the intervals $J_1, \dots, J_{2^k-1}$, respecting the definition of the period doubling pattern.

The construction of a period doubling pattern centered at a fixed point of f , is now complete. ■

Now we demonstrate the above results in the next example:

Suppose that f is the linearization of a periodic orbit of period $24 = 3 \cdot 2^3$. We will demonstrate how the period doubling pattern arises in this case. Assume that the periodic orbit of period 24 is strongly simple, as shown in Figure 12.

Let $\{a_i < b_i < c_i\}_{i=1}^8$ be the 8 blocks of 3 consecutive points. These map monotonically one onto the other with the exception of the last one that maps onto $[a_1, c_1]$ in a non-monotone way. It follows that $f\{a_i, c_i\} = \{a_{i+1}, c_{i+1}\}$ and $f(b_i) = b_{i+1}$ for $i = 1, \dots, 7$. The last block $[a_8, c_8]$ satisfies either $f(a_8) = b_1$, $f(b_8) = c_1$ and $f(c_8) = a_1$ or $f(a_8) = c_1$, $f(b_8) = a_1$ and $f(c_8) = b_1$.

As shown in Figure 12, L maps onto R and viceversa, therefore there exists a fixed point of f in $J_4 = [c_3, a_2]$.

Next we note that LL maps under f^2 onto LR and that RL maps under f^2 onto RR. The intervals $J_2 = [c_5, a_7]$ and $J_6 = [c_6, a_4]$ contain the points of the periodic orbit of period 2 of f .

Similarly we get the points of the orbit of period 4 inside the intervals $J_1 = [c_1, a_5]$, $J_5 = [c_2, a_6]$, $J_3 = [c_7, a_3]$ and $J_7 = [c_4, a_8]$.

The points of the periodic orbit of period 2^j for $j > 2$ are obtained by the period doubling pattern that the first block exhibits under f^8 , and the fact that each block maps onto the next one monotonically, with the exception of the last one.

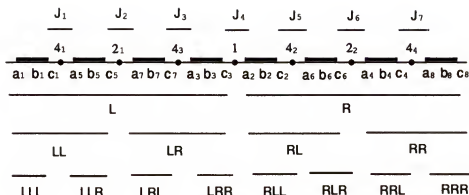


Fig. 12

We now have the machinery to prove Theorem 2.4.2. The proof is an immediate consequence of Theorem 1.0.1 (Sarkovskii's Theorem) and Lemma 2.4.1.

Theorem 2.4.2 *If $f \in C^0(I, I)$ has a periodic point whose period is not a power of 2, then it exhibits a period doubling pattern centered at a fixed point of f .*

Proof: Suppose that f has a periodic point of period $n = q \cdot 2^s$, where $q > 1$ is odd and $s = 0, 1, 2, \dots$. Then by Sarkovskii's Theorem, f has a periodic point of period $3 \cdot 2^{s+1}$. By Lemma 2.4.1, there exists a period doubling pattern centered at a fixed point of f . ■

2.5 Equivalent Condition for the Existence of a Period Doubling Pattern

Let $f \in C^0(I, I)$. The map f will be said to be *turbulent* if there exist compact subintervals J, K with at most one common point such that $J \cup K \subseteq f(J) \cap f(K)$.

Block and Coppel [10](p.26) show that if f is turbulent, then it has periodic points of all periods. In particular, f has points whose periods is not a power of 2 and therefore it is *chaotic*. It follows from the definition that if f is turbulent then f^n is turbulent for every $n > 1$.

A trajectory $\{f^n(c)\}$ will be said to be *alternating* if either $f^k(c) < f^j(c)$ for all even k and all odd j , or $f^k(c) > f^j(c)$ for all even k and all odd j .

A sequence $\{x_k\}$ of real numbers will be said to be *bimonotonic* if for every $m \geq 0$, either $x_k > x_m$ for all $k > m$, or $x_k = x_m$ for all $k > m$, or $x_k < x_m$ for all $k > m$. A trajectory $\{f^n(c)\}$ is *bimonotonic* if and only if there is a z such that the terms $f^k(c) < z$ form an increasing sequence, the terms $f^k(c) > z$ form an decreasing sequence, and $f^k(c) = z$ implies $f^{k+1}(c) = z$.

In the next lemma we require that f^2 is non-turbulent and we characterize the behaviour of the trajectories. For a proof, see [10](p.123).

Lemma 2.5.1 *Let $f \in C^0(I, I)$ and suppose that f^2 is not turbulent.*

Then for each $c \in I$ exactly one of the following alternatives holds:

- (i) *the trajectory $\{f^k(c)\}$ is bimonotonic and converges to a fixed point of f ,*
- (ii) *the trajectory $\{f^k(c)\}$ is alternating from some point on and the limit set $\omega(c, f)$ is an f -orbit of period 2,*

- (iii) *the trajectory $\{f^k(c)\}$ is alternating from some point on and if we set*

$$\alpha = \min \omega(c, f^2), \beta = \max \omega(c, f^2),$$

$$\gamma = \min \omega(f(c), f^2), \delta = \max \omega(f(c), f^2),$$

then $[\alpha, \beta]$ and $[\gamma, \delta]$ are disjoint non-degenerate intervals which contain no fixed point of f .

We will use the following lemma, stated in [10](p.130), to show how the period doubling patterns arise in the case of nonchaotic maps with an infinite limit set.

Lemma 2.5.2 *Suppose that $f \in C^0(I, I)$ is nonchaotic and $\omega(x, f)$ is an infinite limit set for some $x \in I$. For any positive integer s and for any $i = 0, 1, \dots, 2^s - 1$ put $J_i^s = [\inf \omega(f^i(x), f^{2^s}), \sup \omega(f^i(x), f^{2^s})]$.*

Then J_i^s is a compact interval such that $J_i^s \cap J_k^s = \emptyset$ for $0 \leq i < k < 2^s$ and

$$f(J_i^s) \supseteq J_{i+1}^s \text{ for } 0 \leq i < 2^s - 1, \quad f(J_i^s) \supseteq J_0^s \text{ for } i = 2^s - 1.$$

Moreover J_i^s contains a periodic point of period 2^s , but no periodic point of period 2^j with $0 \leq j < s$, and $J_i^{s+1} \cup J_{i+2^s}^{s+1} \subset J_i^s$ for $0 \leq i < 2^s$.

Both J_i^{s+1} and $J_{i+2^s}^{s+1}$ have an endpoint in common with J_i^s .

Proof: Suppose first that $s = 1$, so that $i = 0, 1$.

Then $J_0^1 = [\inf \omega(x, f^2), \sup \omega(x, f^2)]$ and $J_1^1 = [\inf \omega(f(x), f^2), \sup \omega(f(x), f^2)]$. Since f is not chaotic, f^2 is not turbulent. Hence, by Lemma 2.5.1, since $\omega(x, f)$ is an infinite limit set and the alternatives (i) and (ii) of Lemma 2.5.1 are excluded, it follows that J_0^1 and J_1^1 are disjoint non-degenerate intervals which contain no fixed point of f . Moreover $f(J_0^1) \supseteq J_1^1$ and $f(J_1^1) \supseteq J_0^1$. Thus $J_0^1 \rightarrow J_1^1 \rightarrow J_0^1$ and there is a periodic orbit of period 2, denoted by $2_1, 2_2$ as shown in Figure 13 (J_1^1 could lie to the left of J_0^1) :



Fig. 13

Similarly, if $s = 2$ then $i = 0, 1, 2, 3$. Also, $J_0^2 = [\inf \omega(x, f^4), \sup \omega(x, f^4)]$, $J_1^2 = [\inf \omega(f(x), f^4), \sup \omega(f(x), f^4)]$, $J_2^2 = [\inf \omega(f^2(x), f^4), \sup \omega(f^2(x), f^4)]$ and $J_3^2 = [\inf \omega(f^3(x), f^4), \sup \omega(f^3(x), f^4)]$. Moreover, $f(J_0^2) \supseteq J_1^2$, $f(J_1^2) \supseteq J_2^2$, $f(J_2^2) \supseteq J_3^2$ and $f(J_3^2) \supseteq J_0^2$. Finally $J_0^2 \cup J_2^2 \subset J_0^1$ and $J_1^2 \cup J_3^2 \subset J_1^1$, as shown in Figure 14 (J_2^2 could lie to the left of J_0^2 and J_3^2 could lie to the left of J_1^2) :



Fig. 14

Suppose that the lemma holds for all $s = 1, 2, \dots, n$ and all $i = 0, 1, \dots, 2^s - 1$.

We will prove it for $s = n + 1$ and $i = 0, 1, \dots, 2^{n+1} - 1$:

Put $J_i^{n+1} = [\inf \omega(f^i(x), f^{2^{n+1}}), \sup \omega(f^i(x), f^{2^{n+1}})]$.

First note that

$$\omega(x, f) = \omega(x, f^{2^n}) \cup \omega(f(x), f^{2^n}) \cup \dots \cup \omega(f^{2^n-1}(x), f^{2^n}) \text{ and}$$

$$\omega(x, f) = \omega(x, f^{2^{n+1}}) \cup \omega(f(x), f^{2^{n+1}}) \cup \dots \cup \omega(f^{2^{n+1}-1}(x), f^{2^{n+1}}).$$

Also $\omega(f^j(x), f^{2^n}) = \omega(f^j(x), f^{2^{n+1}}) \cup \omega(f^{j+2^n}(x), f^{2^{n+1}})$, for all $0 \leq j \leq 2^n - 1$, so that $J_i^{n+1} \cup J_{i+2^n}^{n+1} \subset J_i^n$ for $0 \leq i < 2^n$.

Finally $f(\omega(x, f^{2^{n+1}})) = \omega(f(x), f^{2^{n+1}})$, so that the intervals J_i^{n+1} are permuted properly according to the induction step. Therefore there is a periodic orbit of period 2^{n+1} , where each point of this orbit is in some J_i^{n+1} for $i = 0, 1, \dots, 2^n - 1$. Also, since $J_i^{n+1} \subset J_i^n$, it contains no periodic point of period 2^j with $0 \leq j < n$. By replacing f^2 with $f^{2^{n+1}}$ in Lemma 2.5.1, we get that J_i^{n+1} contains no periodic point of period 2^n and that the J_i^{n+1} 's are disjoint non-degenerate intervals. This step completes the proof of the Lemma. ■

Corollary 2.5.3 *Let $f \in C^0(I, I)$ be nonchaotic and suppose that $\omega(x, f)$ is an infinite limit set for some $x \in I$. Then there exists a period doubling pattern centered at a fixed point of f .*

Proof: By Lemma 2.5.2, we have points of period 2^j for $j \geq 1$, which respect the definition of a period doubling pattern. We only need to show that there is a fixed point of f according to the definition of the period doubling pattern. Note that the interval with endpoints the two points of period 2, namely $[2_1, 2_2]$ covers itself under f , as shown in Figure 14. Therefore it contains a fixed point of f . By Lemma 2.5.2, the intervals $J_0^1 = [\inf \omega(x, f^2), \sup \omega(x, f^2)]$ and $J_1^1 = [\inf \omega(f(x), f^2), \sup \omega(f(x), f^2)]$ do not contain any fixed point of f . Consequently, the fixed point respects the definition of a period doubling pattern. ■

The following theorem shows that period doubling patterns exist only for maps which are not strongly nonchaotic. We actually prove that there exists a period doubling pattern if and only if the set of periodic points is not closed. But, as mentioned in Chapter 1, one of the characterizations of the strongly nonchaotic maps is that the set of periodic points is closed.

Theorem 2.5.4 *Let $f \in C^0(I, I)$. There is a period doubling pattern centered at a fixed of f if and only if the set of periodic points is not closed.*

Proof: We first suppose that there exists a period doubling pattern centered at a fixed point of f . Choose a sequence p_n , consisting of periodic points of period 2^n for $n = 0, 1, 2, \dots$, where a point belongs to the sequence only if it is part of the period doubling pattern and is the smallest with such period.

Thus $\{p_n\} = p_0, p_1, p_2, \dots$, where p_0 is the fixed point, p_1 is the smallest point of period 2, p_2 is the smallest point of period 4 and in general p_i is the smallest point of period 2^i . By definition $\{p_n\}$ is a monotonically decreasing and bounded sequence in I , and hence it converges.

Let p be the limit of this sequence. We will show by contradiction, that p is not a periodic point. Suppose that p is a periodic point. If the period of p is not a power of 2, then f is a chaotic map, and the set of periodic points is not closed. Therefore, we may assume that the period of p is a power of 2.

By construction, every periodic orbit of the period doubling pattern is simple. Consequently, if (P, π) is a periodic orbit of period 2^n in the period doubling pattern, then the left and right halves of P each form simple orbits of π^2 with period 2^{n-1} . It follows that the image under f of any member of the sequence $\{p_n\}$ for $n \geq 1$, lies to the right of the fixed point p_0 . Thus $f(p_i) > p_0 \forall i \geq 1$, and since p is the limit of the sequence $\{p_i\}$, $f(p) \geq p_0$. Since p is a periodic point itself, it follows that $f(p) > p_0$. Thus p cannot be a fixed point.

Next we show that p cannot have period 2 either. By way of contradiction suppose that p has period 2. Then p is a fixed point for f^2 . The interval $[p, p_0]$ contains properly the fixed point p_1 of f^2 . Moreover, the interval $[p, p_0]$ contains the left half of the f -orbit, which itself is a simple orbit under f^2 . Therefore, $f^2(p_i) > p_1 \forall i \geq 2$, and since p is the limit of the sequence $\{p_i\}$, $f^2(p) \geq p_1$. Since p is a periodic point itself, it follows that $f^2(p) > p_1$. Thus p cannot have period 2.

In the same way, we show that p cannot have period 2^n for all $n \geq 0$. If p has period 2^n then it is a fixed point of f^{2^n} . The interval $[p, p_{n-1}]$ contains properly the fixed point p_n of f^{2^n} . Then, $f^{2^n}(p_i) > p_n \forall i \geq n+1$, and it follows that $f^{2^n}(p) > p_n$. Thus p cannot have period 2^n .

We proved that p is not a periodic point, and hence the set of periodic points is not closed.

Conversely, we suppose that the set of periodic points is not closed.

By [10], this is equivalent to saying that $\omega(x, f)$ is an infinite limit set, for some x . If there exist periodic points whose period is not a power of 2, then by Theorem 2.4.2, f exhibits a period doubling pattern centered at a fixed point. If f is nonchaotic, then by Corollary 2.5.3, we also get a period doubling pattern centered at a fixed point. ■

CHAPTER 3

PERIOD DOUBLING PATTERNS CENTERED AT ORBITS

In this chapter, we investigate period doubling patterns centered at periodic orbits, and the way these arise from chaotic maps on the interval. We assume again, that $f \in C^0(I, I)$ has a periodic orbit whose period is not a power of 2.

The main result of this chapter is Theorem 3.2.8. We show there, that if f has a periodic orbit of period $n > 1$ which is not a power of 2, and $n \prec m$ in Sarkovskii's ordering, then f exhibits a period doubling pattern centered at a periodic orbit of period m .

In Section 3.1, we study θ -adjusted maps and give definitions of tandem cycles, essential fixedpoints and positive and negative representatives. This leads to the definition of positive forcing. Then we define horseshoes, which will help us later to build the period doubling patterns.

In Section 3.2, using results of [25], we show that if a cycle θ forces another cycle π , θ is not an extension of π , and π is not a doubling, then θ forces a 2-horseshoe extension of π .

Next, we prove that if f exhibits θ , and θ satisfies the three conditions of the hypothesis of the previous result, then f exhibits a period doubling pattern centered at the points of some representative of π . Then, we give an example which demonstrates one particular representative of a cycle of period 3, forced by a cycle of period 8, such that no period doubling pattern is centered at the points of this representative. Finally, we prove the main theorem that we already mentioned at the beginning.

In Section 3.3, we give an explicit way of constructing orbits of period $n + 2$, with $n > 1$ odd, forced by orbits of period n . Moreover we show how to construct a period doubling pattern centered at this orbit of period $n + 2$.

In Section 3.4, we give an explicit way of constructing orbits of period 6, starting with any orbit of odd period $n > 1$. Furthermore, we show how to construct a period doubling pattern centered at an orbit of period 6.

The methodology for constructing period doubling patterns, described in Sections 3.3 and 3.4, will evidently generalize for orbits of any period. Thus the constructions of Sections 3.3 and 3.4 could be used to give an alternate proof of Theorem 3.2.8.

3.1 Definitions and Preliminaries

Let $f \in C^0(I, I)$. We have already defined in Section 2.1 what a pattern on a finite f -invariant set $\mathcal{P}$ is. Recall, that we say that the map f exhibits the pattern θ on $\mathcal{P}$ and call $\mathcal{P}$ a *representative* of θ in f .

Also, when $\mathcal{P}$ is a single periodic orbit, the pattern θ represented by $\mathcal{P}$ is a *cycle* (cyclic permutation) and we say that f has a periodic orbit of type θ .

More generally, a bijective pattern θ is a (possibly multicyclic) permutation, and a representative of θ is a finite union of periodic orbits.

Definition 3.1.1 *We say that a pattern θ forces another pattern η , if every continuous map $f : I \rightarrow I$ which exhibits θ also exhibits η .*

We denote by $\mathfrak{B}_n$, $\mathfrak{S}_n$ and $\mathfrak{C}_n$ the set of all patterns, permutations and cycles of length n , respectively; we will drop the subscript to denote the union over all n .

Suppose $f \in C^0(I, I)$ exhibits the pattern $\theta \in \mathfrak{B}_n$ on the finite set $\mathcal{P} = \{x_1 < x_2 < \dots < x_n\}$. By a $\mathcal{P}$ -interval we mean any of the closed intervals $[x_i, x_{i+1}]$ bounded by adjacent elements of $\mathcal{P}$. The map f is $\mathcal{P}$ -linear if $\mathcal{P}$ includes the endpoints of I and f is affine on each $\mathcal{P}$ -interval. When we wish to specify the pattern θ exhibited by f on $\mathcal{P}$, we will call f θ -linear on $\mathcal{P}$. Note that any two θ -linear maps are topologically

conjugate, by [12], and in particular they are all conjugate to the canonical θ -linear map for which $\mathcal{P} = \{1, \dots, n\}$.

Given $f \in C^0(I, I)$, an itinerary of length k is a sequence $\mathcal{I} = \{I_j\}_{j=0}^k$ of $\mathcal{P}$ -intervals; its representative set for f is $\mathcal{R}(f, \mathcal{I}) = \{x | f^j(x) \in I_j \text{ for } j = 0, \dots, k\}$.

We will say $i, i+1 \in \{1, \dots, n\}$ belong to *tandem cycles* in $\theta \in \mathfrak{B}_n$, if $\theta^j(i+1) = \theta^j(i) \pm 1$ for all j , and both $i, i+1$ are cyclic under θ . If $i, i+1$ are tandem and belong to the same cycle of θ , we call this a *self-tandem cycle* in θ .

We will refer to a $\mathcal{P}$ -interval whose endpoints belong to a single self-tandem cycle as a self-tandem interval. It is known [25] that, if f is θ -linear on P and θ has no self-tandem cycles, then any two periodic orbits of f outside P which have the same itinerary represent the same cycle.

Definition 3.1.2 A $\mathcal{P}$ -monotone map f is θ -adjusted on $\mathcal{P}$ provided:

- (i) if $\theta(i) \neq \theta(i+1)$ then f is strictly monotone on the $\mathcal{P}$ -interval $[x_i, x_{i+1}]$;
- (ii) if $x \notin \mathcal{P}$ is periodic with period k , let $\mathcal{I}(x) = \{I_j\}_{j=0}^k$ be the unique k -loop with $x \in \mathcal{R}(f, \mathcal{I}(x))$. Then $\mathcal{R}(f, \mathcal{I}(x))$ contains no other fixed point of f^k .

The proof of the following lemma due to Misiurewicz and Nitecki [25], but it is worthwhile to repeat it here, since it describes the construction of a specific θ -adjusted map.

The specific θ -adjusted map produced by the construction in the following proof, starting from the $\mathcal{P}$ -linear map with $\mathcal{P} = \{1, \dots, n\}$, will be called the *canonical θ -adjusted map*.

Lemma 3.1.3 For any $\mathcal{P}$ -linear map f , there exists a $\mathcal{P}$ -adjusted map F which agrees with f on any $\mathcal{P}$ -interval whose endpoints do not belong to tandem cycles.

Proof: Let $\mathcal{P}_{\frac{1}{2}} \supset \mathcal{P}$ be the union of $\mathcal{P}$ with the midpoints of all self-tandem $\mathcal{P}$ -intervals. Note first that if F is $\mathcal{P}_{\frac{1}{2}}$ -adjusted then it is automatically $\mathcal{P}$ -adjusted; so we

work with $\mathcal{P}_{\frac{1}{2}}$ in place of $\mathcal{P}$. Also up to conjugacy, we can work with $\mathcal{P}_{\frac{1}{2}} = \{1, \dots, n\}$.

Construct F as follows:

(i) On any $\mathcal{P}_{\frac{1}{2}}$ -interval $J = [i, i+1]$, $f = F$ unless $i, i+1$ belong to tandem cycles in $\theta_{\frac{1}{2}}$ and $i \leq \theta^j(i), \theta^j(i+1)$, for all j .

(ii) If $i, i+1$ belong to tandem cycles in $\theta_{\frac{1}{2}}$ and i is the leftmost point in the union of these cycles, then the period k of i is at least equal to the period of $i+1$; since $\theta_{\frac{1}{2}}$ has no self-tandem cycles, we have $f^j(intJ)$ disjoint for $j = 0, \dots, k-1$, and $f^k = id.$ on J . For $0 \leq t \leq 1$, let $F(i+t) = f(i+t^2)$.

Then we see that on any $\mathcal{P}_{\frac{1}{2}}$ -interval $J = [i, i+1]$ bounded by tandem cycles, $F^k(i+t) = i+t^2$. Hence, no periodic orbit of F intersect $intJ$. It follows that F is $\mathcal{P}$ -adjusted. ■

Misiurewicz and Nitecki, proved [25], that if $\theta \in \mathfrak{C}$ and F is θ -adjusted, then $\eta \in \mathfrak{B}$ is forced by θ if and only if η is exhibited by F .

Definition 3.1.4 Given $f \in C^0(I, I)$ and $x \in I$ with $f(x) = x$, we say that x is an essential fixedpoint of f with index $\sigma(f, x) = \sigma \in \{\pm 1\}$ if for some $\epsilon > 0$ we have

$$\text{sign}[f(y) - y] = \begin{cases} -\sigma & \text{for } x - \epsilon < y < x \\ \sigma & \text{for } x < y < x + \epsilon. \end{cases}$$

We will call x a positive (resp. negative) fixedpoint if it is essential with positive (resp. negative) index.

Note that a fixedpoint is essential if and only if it is an isolated fixedpoint and the graph of f crosses the diagonal at (x, x) . Essential fixedpoints are precisely the points where $f(y) - y$ switches sign.

Definition 3.1.5 Let $Q = \{x_1, \dots, x_k\}$ be a representative of $\eta \in \mathfrak{C}_k$ in a piecewise monotone map f of the interval I and assume that x_1, x_k are not endpoints of I . We will call Q a positive (resp. negative) representative of η in f if $\sigma(f^k, x) = +1$ (resp. -1) at every $x \in Q$.

Definition 3.1.6 A pattern $\theta \in \mathfrak{B}$ positively forces $\pi \in \mathfrak{C}$ if a θ -adjusted map has a positive representative of π .

Definition 3.1.7 Given $k \geq 2$ and $\sigma \in \{\pm 1\}$, the k -horseshoe with sign σ is the pattern $h(k, \sigma) \in \mathfrak{B}_{k+1}$ defined by

$$h(k, \sigma)(i) = \begin{cases} 1 & \text{if } (-1)^{i+1} = \sigma \\ k+1 & \text{if } (-1)^{i+1} = -\sigma \end{cases}$$

for $i = 1, \dots, k+1$.

As an example, we show the 2-horseshoe with $\sigma = 1$, in Figure 15:

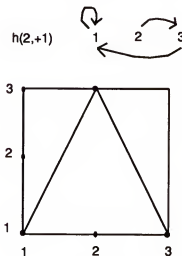


Fig. 15

Remark: let $I_1 = [1, 2]$ and $I_2 = [2, 3]$. Then both I_1 and I_2 cover the whole interval $I_1 \cup I_2 = [1, 3]$. The loop $I_1 \rightarrow I_2 \rightarrow I_2 \rightarrow I_1$ has length 3 and it is not repetitive.

Therefore $h(2, +1)$ always forces a periodic orbit of period 3. Evidently, the same holds for $h(2, -1)$.

Let $\mathcal{P} = \{1, \dots, n\}$. By a block in $\mathcal{P}$, we mean a set of the form $B = \{i \in \mathcal{P} | a \leq i \leq b\}$, where $a \leq b \in \mathcal{P}$.

Suppose $\theta \in \mathfrak{B}$. By a block structure for θ , we mean a partition $\{B_1, \dots, B_k\}$ of $\mathcal{P}$ into disjoint blocks such that if $x, y \in \mathcal{P}$ belong to the same block, their images under θ belong to a single block. We number the blocks so that $x \in B_i, y \in B_j$ and $i < j$ implies that $x < y$; then there is a unique pattern $\eta \in \mathcal{B}_k$ defined by $\theta[B_j] \subset B_{\eta(j)}, j = 1, \dots, k$.

We call η a *reduction* of θ and θ an *extension* of η and refer to $\{B_1, \dots, B_k\}$ as a block structure for θ over η .

In the case of cycles we introduce a more precise terminology : if $\theta \in \mathfrak{C}_{mk}$ is an extension of $\eta \in \mathfrak{C}_k$, we call θ an *m-extension* of η and η an *m-reduction* of θ . As we mentioned in Section 2.1, a 2-extension of a cycle is also called a *doubling* of it. If P is a representative of θ , and Q a representative of η , then we also say that P is an *m-extension* of Q and Q is an *m-reduction* of P .

We will call $\eta \in \mathfrak{B}$ (resp. $\mathfrak{S}, \mathfrak{C}$) a subpattern (resp. subpermutation, subcycle) of $\theta \in \mathfrak{B}$ if η is exhibited by the restriction of θ to some subset of $\{1, \dots, n\}$.

We recall once more the definition of the period doubling pattern centered at a periodic orbit.

Definition 3.1.8 Let $f \in C^0(I, I)$. Suppose that f has a periodic orbit P of period $n \geq 1$ in I . Let $\mathcal{G}$ be a countable collection of points consisting of one periodic orbit P_k of period $2^k n$ for each $k \geq 0$, satisfying the following three conditions:

(i) $P_0 = P$.

(ii) If $p_1 < p_2 < \dots < p_{2^{k+1}n}$ is the periodic orbit P_{k+1} period $2^{k+1}n$, then between p_i and p_{i+1} , where $1 \leq i < 2^{k+1}n$ and i is odd, there exists a unique point z_i of P_k and no other point of period less than or equal to $2^k n$ of $\mathcal{G}$.

(iii) If p_i and p_{i+1} , where i is odd, are any two adjacent points of P_{k+1} and z_i is the unique point of P_k in $[p_i, p_{i+1}]$, then there exists j odd such that $\{f(p_i), f(p_{i+1})\} = \{p_j, p_{j+1}\}$ and $f(z_i) \in (p_j, p_{j+1})$.

Then we say that f exhibits a period doubling pattern centered at P .

3.2 Horseshoe Extensions and Period Doubling Patterns

We begin with two propositions that are proved in [25].

Proposition 3.2.1 *Suppose f is θ -adjusted for $\theta \in \mathfrak{B}$, and let $\pi \in \mathfrak{C}_k$ be forced by θ .*

(i) if π is not a subcycle of θ then every representative of π is essential.

(ii) if

(1) $k > 1$

(2) every π -representative in f is essential and

(3) π has no 2-reductions

then the number of positive π -representatives in f is nonzero.

Proposition 3.2.2 *If $\theta \in \mathfrak{B}_n$ positively forces $\pi \in \mathfrak{C}_k$, and no subcycle of θ extends π , then θ forces extensions of π by $h(2, \sigma)$ for $\sigma = \pm 1$.*

The following proposition gives conditions under which a cycle forces a 2-horseshoe extension of another cycle. This will lead to a useful device for obtaining period doubling patterns.

Proposition 3.2.3 *Let θ and π be in $\mathfrak{C}$ such that :*

(i) θ forces π

(ii) θ is not an extension of π

(iii) π is not a doubling.

Then θ forces extensions of π by $h(2, \sigma)$ for $\sigma = \pm 1$.

Proof: Since θ and π are cycles such that θ is not an extension of π , they must be of different type. Therefore, they are not equal, and in particular π is not a subcycle of θ .

Let f be any θ -adjusted map. It follows by Proposition 3.2.1 (i), that every representative of π is essential. Also, since θ is not an extension of π , the length of π is greater than one. Finally, since π is not a doubling, by Proposition 3.2.1 (ii), we obtain that the number of positive π -representatives in f is nonzero.

This means that θ positively forces π , and since θ is not an extension of π , no subcycle of θ extends π . By Proposition 3.2.2, θ forces extensions of π by $h(2, \sigma)$ for $\sigma = \pm 1$. ■

The following theorem is proved in [10](p.171):

Theorem 3.2.4 *Let P and Q be periodic orbits of different types. Then P forces Q if and only if the linearization of P has an orbit of the same type as Q .*

Before we show how we obtain period doubling patterns centered at periodic orbits using 2-horseshoe extensions, we state a lemma whose proof is given in [9]. This will enable us to prove Theorem 3.2.7.

Lemma 3.2.5 *Let $f \in C^0(I, I)$ have a periodic orbit $P = \{x_1 < \dots < x_n\}$ of period $n = 3m$, where $m > 1$. Suppose that f cyclically permutes the sets $\alpha_k = \{x_{3(k-1)+1}, x_{3(k-1)+2}, x_{3k}\}$ ($k = 1, \dots, m$).*

Then f has a periodic orbit $Q = \{y_1 < \dots < y_n\}$ of period n such that f cyclically permutes the sets $\beta_k = \{y_{3(k-1)+1}, y_{3(k-1)+2}, y_{3k}\}$ ($k = 1, \dots, m$) and in addition:

- (i) $\beta_k \subseteq [x_{3(k-1)+1}, x_{3k}]$ for $k = 1, \dots, m$,
- (ii) f maps β_k monotonically onto $f(\beta_k)$ for all but one k .

The following proposition, generalizes Proposition 2.3.2.

Proposition 3.2.6 *Let θ and $\pi \in \mathfrak{C}$ such that θ forces π , and θ is not an extension of π . Let $f \in C^0(I, I)$ exhibit θ on a periodic orbit P . Let g denote the linearization of θ on P . If g exhibits a period doubling pattern centered at a representative of π , then f exhibits a period doubling pattern centered at a representative of π .*

Proof: Let Q denote a representative of π , such that g exhibits a period doubling pattern centered on Q . Since θ forces π , f exhibits a periodic orbit of the same type as Q , which we call R .

Let m be the period of Q . Let $\{Q_j\}_{j=0}^{\infty}$ denote the collection of periodic orbits of the period doubling pattern of g centered at Q , where Q_j is a periodic orbit of period $2^j m$, for each $j \geq 0$.

We will construct a period doubling pattern $\{R_j\}_{j=0}^{\infty}$ centered at R , using induction. First we set $R_0 = R$, and we can start the induction. Assume that $\{R_j\}_{j=0}^N$ has been constructed, so that $P \cup Q_0 \cup Q_1 \cup \dots \cup Q_N$ and $P \cup R_0 \cup R_1 \cup \dots \cup R_N$ have the same type. For each non-negative integer i , let $A_i = \{x : g^i(x) \in P \cup Q_0 \cup Q_1 \cup \dots \cup Q_N\}$. Note that $A_i = g^{-1}(A_{i-1})$ and A_i is finite for each positive integer i .

We claim that each interval joining adjacent points of $A_{2^{N+1}m}$ contains at most one point of Q_{N+1} . To see this, first note that the orbits P and Q_{N+1} are of different type, since P is not an extension of Q , while Q_{N+1} is an extension of Q .

Now let $\tilde{A}_n = \{x : g^n(x) \in P\}$. It follows from Lemma 2.3.1 applied to P and Q_{N+1} that each interval joining adjacent points of $\tilde{A}_{2^{N+1}m}$ contains at most one point of Q_{N+1} . Since $\tilde{A}_{2^{N+1}m} \subseteq A_{2^{N+1}m}$, each interval joining adjacent points of $A_{2^{N+1}m}$ is contained in some interval joining adjacent points of $\tilde{A}_{2^{N+1}m}$. Hence, we conclude that each interval joining adjacent points of $A_{2^{N+1}m}$ contains at most one point of Q_{N+1} .

Now let $B_0 = P \cup R_0 \cup R_1 \cup \dots \cup R_N$, so that B_0 and A_0 are of the same type. We define $B_1, B_2, \dots, B_{2^{N+1}m}$ inductively as follows. Assuming B_{n-1} has been defined, we take B_n to be a subset of $f^{-1}(B_{n-1})$ such that:

- (i) $B_n \supseteq B_{n-1}$,
- (ii) B_n and A_n have the same cardinality,
- (iii) if $A_n = \{y_1 < \dots < y_n\}$ and $B_n = \{z_1 < \dots < z_n\}$, then $g|_{A_n}$ and $f|_{B_n}$

have the same type.

The finite sets $A_{2^{N+1}m}, B_{2^{N+1}m}$ are invariant under g, f respectively, and the Markov graphs of $(A_{2^{N+1}m}, g)$ and $(B_{2^{N+1}m}, f)$ are the same. By following the orbit of Q_{N+1} , we obtain a loop in the Markov graph of $(A_{2^{N+1}m}, g)$. This same loop in the

Markov graph of $(B_{2^{N+1}m}, f)$ yields a periodic orbit R_{N+1} of f . Note that each interval joining adjacent points of $B_{2^{N+1}m}$ contains at most one point of R_{N+1} . Therefore $g|_{P \cup Q_0 \cup Q_1 \cup \dots \cup Q_{N+1}}$ and $f|_{P \cup R_0 \cup R_1 \cup \dots \cup R_{N+1}}$ have the same type, and $R_0 \cup R_1 \cup \dots \cup R_{N+1}$ respects the definition of a period doubling pattern centered at R . The induction step is now complete. ■

Theorem 3.2.7 *Let $f \in C^0(I, I)$ exhibit a cyclic permutation $\theta \in \mathfrak{C}$. Suppose that $\pi \in \mathfrak{C}_m$, $(m > 1)$, such that*

- (i) θ forces π
- (ii) θ is not an extension of π
- (iii) π is not a doubling.

Then f exhibits a period doubling pattern centered at the points of some representative of π in f .

Proof: First notice that, by Proposition 3.2.3, θ forces extensions of $\pi \in \mathfrak{C}_m$, $(m > 1)$, by $h(2, \sigma)$, for $\sigma = \pm 1$. In the remark following the definition of a k -horseshoe, in Section 3.1, we showed that a 2-horseshoe forces a periodic orbit of period 3. Therefore, by following the Markov graph of f , we obtain a nonrepetitive loop of length $3m$. Hence, f has a periodic orbit of period $3m$, which we denote by $P = \{x_1 < \dots < x_{3m}\}$. By construction, f cyclically permutes the m sets $\alpha_k = \{x_{3(k-1)+1}, x_{3(k-1)+2}, x_{3k}\}$ ($k = 1, \dots, m$).

By Lemma 3.2.5, f has a periodic orbit $Q = \{y_1 < \dots < y_{3m}\}$ of period $3m$, such that f cyclically permutes the sets $\beta_k = \{y_{3(k-1)+1}, y_{3(k-1)+2}, y_{3k}\}$ ($k = 1, \dots, m$) and in addition:

- (i) $\beta_k \subseteq [x_{3(k-1)+1}, x_{3k}]$ for $k = 1, \dots, m$,
- (ii) f maps β_k monotonically onto $g(\beta_k)$ for all but one k .

Let g denote the linearization of Q . By renumbering the β_k 's if necessary we may assume that g maps β_k monotonically onto β_{k+1} for $k = 1, \dots, m-1$, and β_m in a non-monotone way onto β_1 .

Notice that g^m has a periodic orbit of period 3, in the convex hull of β_1 . Then, by Lemma 2.2.1, applied to g^m , we obtain that g^m exhibits a period doubling pattern centered at a fixed point of g^m , in the convex hull of β_1 . Let 1 denote this fixed point. The blocks β_k ($k = 1, \dots, m$), are cyclically permuted, and hence $g^k(1)$ is in the convex hull of β_{k+1} for $k = 1, \dots, m-1$. Thus the orbit of 1 under g is a periodic orbit of period m .

Next, let $2_1, 2_2$ denote the points of period 2, under g^m , of the period doubling pattern in β_1 . Therefore $g^m(2_1) = 2_2$ and $g^m(2_2) = 2_1$. By linearity of g , and since each β_k maps monotonically onto β_{k+1} for $k = 1, \dots, m-1$, the points $g^k(2_1)$ and $g^k(2_2)$, lie in the convex hull of β_{k+1} for $k = 1, \dots, m-1$, in a way that they respect the definition of the period doubling pattern. Thus the orbit of 2_1 under g is a periodic orbit of period $2m$.

Proceeding in this way, we obtain periodic orbits of period $2^i m$, for all $i \geq 0$, in a way that they respect the definition of the period doubling pattern. Hence g exhibits a period doubling pattern centered at a periodic orbit of period m , having the type of π .

By Proposition 3.2.6, it follows that f exhibits a period doubling pattern centered at the points of some representative of π in f . ■

The following example, stresses the conclusion of Theorem 3.2.7, in the sense, that f does not exhibit a period doubling pattern centered at all representatives of π .

Let f be the linearization of $\theta \in \mathfrak{C}_8$ on the orbit given in Figure 16:

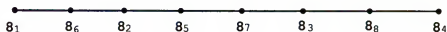


Fig. 16

As usual, we assume that $f(8_i) = 8_{i+1}$ ($i = 1, \dots, 7$) and $f(8_8) = 8_1$.

Then f exhibits $\pi \in \mathcal{C}_3$ on the orbit determined by the loop

$$[8_6, 8_2] \rightarrow [8_7, 8_3] \rightarrow [8_8, 8_4] \rightarrow [8_6, 8_2],$$

as shown in Figure 17:

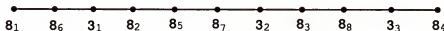


Fig. 17

Again, we assume that $f(3_j) = 3_{j+1}$ ($j = 1, 2$) and $f(3_3) = 3_1$.

Notice that θ forces π , θ is not an extension of π and π is not a doubling. Therefore, by Theorem 3.2.7, f exhibits a period doubling pattern centered at the points of some representative of π , which is a periodic orbit of period 3.

Nevertheless, we will show that f does not exhibit any period doubling pattern centered at the given periodic orbit of period 3. This does not contradict Theorem 3.2.7, since the conclusion of the theorem is that f exhibits a period doubling pattern centered at the points of some representative of π , which in our case, is another orbit of period 3 of the type of π .

It suffices to show that, there is no periodic orbit of period 6, belonging to a possible period doubling pattern centered at this periodic orbit of period 3.

If such an orbit exists then the interval $[3_3, 8_4]$ must contain a point of this orbit. Suppose 6_1 belongs to this interval. Then 6_2 belongs to either $[3_1, 8_2]$ or $[8_2, 8_5]$. Since $[3_1, 8_2] \rightarrow [3_2, 8_3] \rightarrow [3_3, 8_4]$, 6_2 cannot belong to $[3_1, 8_2]$ and therefore $6_2 \in [8_2, 8_5]$.

Now 6_6 belongs to either $[3_2, 8_3]$ or $[8_3, 8_8]$. In any case $6_6 > 3_2$. Since 3_1 is the unique point of period 3 of the period doubling pattern, that lies between 6_2 and 6_5 , and $6_2 > 3_1$, it follows that $6_6 < 3_1$. But no point less than 3_1 can map onto 6_6 . This is

a contradiction, and hence f does not exhibit a period doubling pattern centered on this orbit of period 3.

Now, we consider Sarkovskii's ordering, and give conditions for the existence of a period doubling pattern centered at a periodic orbit.

Theorem 3.2.8 *Let $f \in C^0(I, I)$. Suppose that f has a periodic orbit of period $n > 1$ which is not a power of 2. If $n \prec m$ in Sarkovskii's ordering, then f exhibits a period doubling pattern centered at a periodic orbit of period m .*

Proof: We consider two different cases :

(i) $m \neq 2^s$ for $s = 0, 1, 2, \dots$

Suppose that f exhibits θ on the periodic orbit of period n . Let L denote the linearization of θ .

Since $n \prec m$, it follows by Sarkovskii's Theorem that L has a periodic orbit of period m . Also by Theorem 2.1.8, L has a periodic orbit of period m which is strongly simple and therefore not a doubling.

Suppose that L exhibits π on the strongly simple periodic orbit of period m . We claim that, since $n \prec m$, the orbit of period n cannot be an extension of the orbit of period m . If $n < m$, this is evident. If $n > m$, then $n = 2^{k_1} q_1$ and $m = 2^{k_2} q_2$, for some odd integers $q_1, q_2 > 1$, and nonnegative integers k_1, k_2 such that $k_1 < k_2$. Then, if there exists a positive integer λ such that $n = \lambda m$, we obtain that $2^{k_1} q_1 = \lambda(2^{k_2} q_2)$. It follows that $q_1 = \lambda(2^{k_2 - k_1} q_2)$, which is a contradiction, since the left hand side of the equality is odd and the right is even.

Hence θ forces π , θ is not an extension of π and π is not a doubling. Thus, by Theorem 3.2.7, f exhibits a period doubling pattern centered at the points of a representative of π , which is a periodic orbit of period m .

(ii) $m = 2^s$ for $s = 0, 1, 2, \dots$

By Theorem 2.4.2, f exhibits a period doubling pattern centered at a fixed point of f . We remove all the periodic orbits of period 2^j for all $j = 0, 1, \dots, s - 1$

and we are left with a period doubling pattern centered at a periodic orbit of period m . ■

3.3 Construction of Patterns Centered at Orbits of Odd Period

We have seen that if a continuous function f has a periodic orbit of odd period $n > 1$, then it has a period doubling pattern centered at a periodic orbit of period $n + 2$. Indeed, this follows from Theorem 3.2.8.

In the next lemma, we explicitly show how we can construct such a period doubling pattern. In this process we show a systematic way of getting a periodic orbit of period $n + 2$, given a periodic orbit of odd period n .

Lemma 3.3.1 *Suppose that $f \in C^0(I, I)$ has a periodic orbit of odd period $n > 1$. Then f exhibits a period doubling pattern centered at a periodic orbit of period $n + 2$. Furthermore, f exhibits a period doubling pattern centered at a periodic orbit of period $n + m$, for all even $m \geq 2$.*

Proof: The existence of a periodic orbit of period $n + 2$ follows by Theorem 1.0.1 (Sarkovskii's theorem). We will show how we obtain such a periodic orbit and then how we construct a period doubling pattern centered at this orbit.

Since f has a periodic orbit of odd period $n > 1$, we can assume by Proposition 2.1.5 that it is of Stefan type and it has the configuration shown in Figure 18. Also since $[(n)_2, (n)_3] \rightarrow [(n)_2, (n)_3]$ there is a fixed point in $[(n)_2, (n)_3]$, which we denote by 1 :

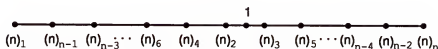


Fig. 18

Using the points $(n)_i$ ($i = 1, \dots, n$) and the fixed point 1, we construct a nonrepetitive loop of length $n - 1$, as follows :

$$[(n)_1, (n)_{n-1}] \rightarrow [1, (n)_3] \rightarrow [(n)_4, (n)_2] \rightarrow [(n)_3, (n)_5] \rightarrow [(n)_6, (n)_4] \rightarrow \dots \rightarrow [(n)_{n-4}, (n)_{n-2}] \rightarrow [(n)_{n-1}, (n)_{n-3}] \rightarrow [(n)_{n-2}, (n)_n] \rightarrow [(n)_1, (n)_{n-1}].$$

Hence there exists a periodic orbit of period $n - 1$, as shown in Figure 19:

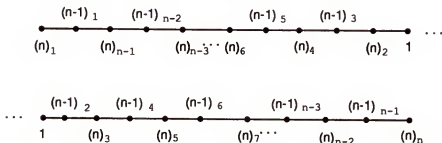


Fig. 19

Now, using the points $1, (n)_i$ ($i = 1, \dots, n$) and $(n-1)_j$ ($j = 1, \dots, n-1$), we construct a nonrepetitive loop of length $n+1$, as follows :

$$[(n)_1, (n-1)_1] \rightarrow [1, (n-1)_2] \rightarrow [(n)_2, 1] \rightarrow [(n-1)_2, (n)_3] \rightarrow [(n)_4, (n-1)_3] \rightarrow [(n-1)_4, (n)_5] \rightarrow \dots \rightarrow [(n)_{n-1}, (n-1)_{n-2}] \rightarrow [(n-1)_{n-1}, (n)_n] \rightarrow [(n)_1, (n-1)_1].$$

Note that for the construction of the above loop, we do not use the intervals $[(n-1)_1, (n)_{n-1}]$, $[(n-1)_{2k-1}, (n)_{2k-2}]$ and $[(n)_{2k-1}, (n-1)_{2k}]$ for $k = 2, \dots, \frac{n-1}{2}$.

Hence there exists a periodic orbit of period $n+1$, as shown in Figure 20:

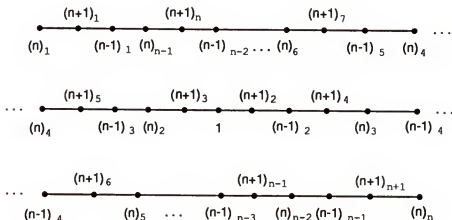


Fig. 20

We are now ready to exhibit the periodic orbit of period $n + 2$. Using the points $1, (n)_i$ ($i = 1, \dots, n$), $(n - 1)_j$ ($j = 1, \dots, n - 1$) and $(n + 1)_l$ ($l = 1, \dots, n + 1$) we form the following nonrepetitive loop of length $n + 2$:

$$\begin{aligned} &[(n)_1, (n + 1)_1] \rightarrow [(n + 1)_3, 1] \rightarrow [(n + 1)_2, (n - 1)_2] \rightarrow [(n - 1)_3, (n)_2] \rightarrow \\ &[(n)_3, (n - 1)_4] \rightarrow [(n - 1)_5, (n)_4] \rightarrow [(n)_5, (n - 1)_6] \rightarrow \dots \rightarrow [(n - 1)_{n-2}, (n)_{n-3}] \rightarrow \\ &[(n)_{n-2}, (n - 1)_{n-1}] \rightarrow [(n - 1)_1, (n)_{n-1}] \rightarrow [(n + 1)_{n+1}, (n)_n] \rightarrow [(n)_1, (n + 1)_1]. \end{aligned}$$

Hence there exists a periodic orbit of period $n + 2$, as shown in Figure 21:

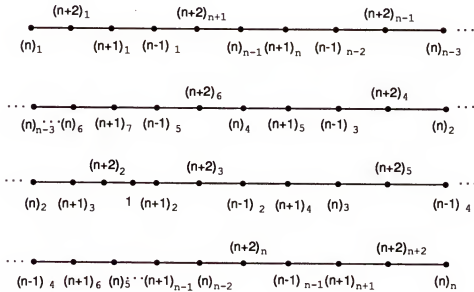


Fig. 21

We are now ready to show that f exhibits a period doubling pattern centered at this periodic of period $n + 2$. We start by showing that there exists a periodic orbit of period $2(n + 2)$. The algorithm is as follows:

We start with the very left interval of Figure 21, which is $[(n)_1, (n + 2)_1]$, and follow the orbit of $(n + 2)_1$ up to the point $(n + 2)_{n+2}$. Each time we choose the interval with endpoints $(n + 2)_i$ ($i = 1, \dots, n + 2$) and the adjacent point to $(n + 2)_i$ covered by the previous interval in the loop. Note that in this way we obtain the half of the required loop and that the last interval $[(n + 1)_{n+1}, (n + 2)_{n+2}]$ covers

$[(n+2)_1, (n+1)_1]$. Again, we follow the orbit of $(n+2)_1$, but now using the intervals to the opposite side of $(n+2)_1$, up to the point $(n+2)_3$. Next we follow the orbit of the point $(n)_2$ up to the point $(n)_n$. Each time we choose the interval with endpoints $(n)_j$ ($j = 2, \dots, n$) and the adjacent point to $(n)_j$ covered by the previous interval in the loop. Finally $[(n+2)_{n+2}, (n)_n]$ covers $[(n)_1, (n+2)_1]$ and the loop is complete:

$$\begin{aligned}
&[(n)_1, (n+2)_1] \rightarrow [(n+1)_3, (n+2)_2] \rightarrow [(n+2)_3, (n-1)_2] \rightarrow \\
&[(n-1)_3, (n+2)_4] \rightarrow [(n+2)_5, (n-1)_4] \rightarrow \\
&[(n-1)_5, (n+2)_6] \rightarrow [(n+2)_7, (n-1)_6] \rightarrow \\
&[(n-1)_7, (n+2)_8] \rightarrow [(n+2)_9, (n-1)_8] \rightarrow \dots \rightarrow \\
&[(n-1)_{n-2}, (n+2)_{n-1}] \rightarrow [(n+2)_n, (n-1)_{n-1}] \rightarrow [(n-1)_1, (n+2)_{n+1}] \rightarrow \\
&[(n+1)_{n+1}, (n+2)_{n+2}] \rightarrow [(n+2)_1, (n+1)_1] \rightarrow [(n+2)_2, 1] \rightarrow \\
&[(n+1)_2, (n+2)_3] \rightarrow [(n)_2, (n+1)_3] \rightarrow \\
&[(n+1)_4, (n)_3] \rightarrow [(n)_4, (n+1)_5] \rightarrow \\
&[(n+1)_6, (n)_5] \rightarrow [(n)_6, (n+1)_7] \rightarrow \\
&[(n+1)_8, (n)_7] \rightarrow [(n)_8, (n+1)_9] \rightarrow \dots \rightarrow \\
&[(n+1)_{n-1}, (n)_{n-2}] \rightarrow [(n)_{n-1}, (n+1)_n] \rightarrow \\
&[(n+2)_{n+2}, (n)_n] \rightarrow [(n)_1, (n+2)_1].
\end{aligned}$$

It is clearly a nonrepetitive loop, and furthermore there are intervals in this loop that are not adjacent. This guarantees that there exists a periodic orbit of period $2(n+2)$. Moreover, the periodic orbit of period $2(n+2)$ respects the definition of the period doubling pattern, as shown in Figure 22:

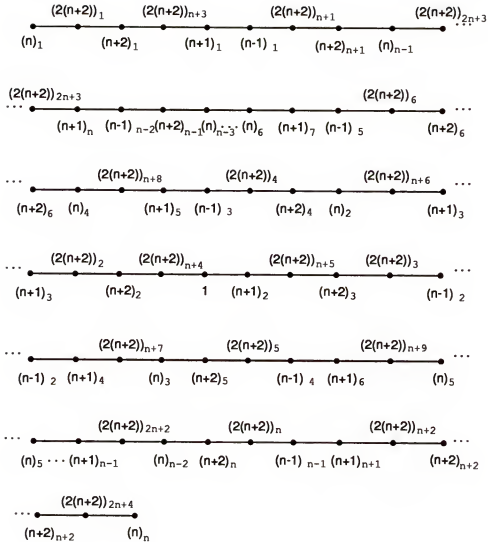


Fig. 22

Now using induction we show that f exhibits a period doubling pattern centered at the periodic orbit of period $n+2$. We already proved it for the periodic orbit of period $2(n+2)$. Assume that we have constructed the periodic orbits of period up to $2^k(n+2)$, so that they respect the definition of the period doubling pattern centered at the periodic orbit of period $n+2$.

We will describe how we obtain a periodic orbit of period $2^{k+1}(n+2)$, respecting the period doubling pattern centered at the periodic orbit of period $n+2$. The

corresponding loop is obtained as follows:

We start with the interval $[(n)_1, (2^k(n+2))_1]$, and follow the orbit of $(2^k(n+2))_1$ up to the point $(2^k(n+2))_{2^k(n+2)}$. Each time we choose the interval with endpoints $(2^k(n+2))_i$ ($i = 1, \dots, 2^k(n+2)$) and the adjacent point to $(2^k(n+2))_i$ covered by the previous interval in the loop. Next we continue following the orbit of $(2^k(n+2))_1$, but now the intervals chosen, are the ones lying to the opposite side of $(2^k(n+2))_i$, for $i = 1, \dots, 2^k(n+2) - (n-1)$. Then, as in the case of the orbit of period $2(n+2)$, we follow the orbit of the point $(n)_2$ up to the point $(n)_n$. Each time we choose the interval with endpoints $(n)_j$ ($j = 2, \dots, n$) and the adjacent point to $(n)_j$ covered by the previous interval in the loop. Finally $[(2^k(n+2))_{2^k(n+2)}, (n)_n]$ covers $[(n)_1, (2^k(n+2))_1]$ and the loop is complete. It follows that the loop obtained in this way is nonrepetitive, and by construction respects the period doubling pattern centered at the periodic orbit of period $n+2$. This completes the induction step.

To finish the proof of the lemma, we observe that by Sarkovskii's ordering, if f has a periodic orbit of odd period $n > 1$, then it has periodic orbits of all odd periods $n+m-2$ with even $m \geq 2$. Hence, by the first part of this lemma, f has a period doubling pattern centered at a periodic orbit of period $n+m$ for all $m \geq 2$.

■

3.4 More on the Construction of Period Doubling Patterns

In the next lemma, we explicitly show how to construct a period doubling pattern centered at a periodic orbit of period 6, given a periodic orbit of odd period $n > 1$.

Lemma 3.4.1 *Suppose that $f \in C^0(I, I)$ has a periodic orbit of odd period $n > 1$. Then f exhibits a period doubling pattern centered at a periodic orbit of period 6.*

Proof: The existence of a periodic orbit of period 6 follows by Theorem 1.0.1 (Sarkovskii's theorem). We will show how we obtain such a periodic orbit and then how we construct a period doubling pattern centered at this orbit.

Since f has a periodic orbit of odd period $n > 1$, we can assume by Proposition 2.1.5 that it is of Stefan type. Also, without loss of generality suppose that $n \geq 7$. Let $(n)_1 < (n)_{n-1} < (n)_{n-3} < (n)_{n-5} < \dots < (n)_{n-6} < (n)_{n-4} < (n)_{n-2} < (n)_n$ be the points of the periodic orbit of period n .

Then, there exists a loop of length 6, given by

$$\begin{aligned} [(n)_1, (n)_{n-1}] &\rightarrow [(n)_{n-6}, (n)_{n-4}] \rightarrow [(n)_{n-3}, (n)_{n-5}] \rightarrow [(n)_{n-4}, (n)_{n-2}] \rightarrow \\ &[(n)_{n-1}, (n)_{n-2}] \rightarrow [(n)_{n-2}, (n)_n] \rightarrow [(n)_1, (n)_{n-1}]. \end{aligned}$$

Hence f has a periodic orbit of period 6, which is strongly simple, denoted by $6_1, \dots, 6_6$, as shown in Figure 23:

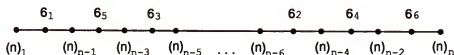


Fig. 23

We are now ready to show that f exhibits a period doubling pattern centered at this periodic of period 6.

First, there is a loop of length 12, given below :

$$\begin{aligned} [(n)_1, 6_1] &\rightarrow [(n)_{n-6}, 6_2] \rightarrow [6_3, (n)_{n-5}] \rightarrow [(n)_{n-4}, 6_4] \rightarrow \\ [6_5, (n)_{n-3}] &\rightarrow [(n)_{n-2}, 6_6] \rightarrow [6_1, (n)_{n-1}] \rightarrow [(n)_{n-4}, 12_4] \rightarrow \\ [12_5, (n)_{n-3}] &\rightarrow [(n)_{n-2}, 12_6] \rightarrow [12_7, (n)_{n-1}] \rightarrow [6_6, (n)_n] \rightarrow [(n)_1, 6_1]. \end{aligned}$$

Thus there exists a periodic orbit of period 12, denoted by $12_1, \dots, 12_{12}$, and respecting the definition of the period doubling pattern, as shown in Figure 24:

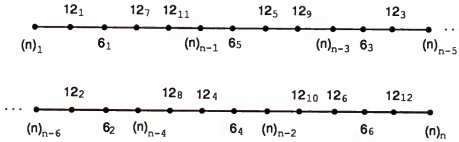


Fig. 24

Note that the intervals $[6_2, (n)_{n-4}]$, $[(n)_{n-3}, 6_3]$, $[6_4, (n)_{n-2}]$ and $[(n)_{n-1}, 6_5]$ do not contain any point of period 12.

Continuing in the same way, we obtain a loop of length 24, as follows :

$$\begin{aligned}
 &[(n)_1, 12_1] \rightarrow [(n)_{n-6}, 12_2] \rightarrow [12_3, (n)_{n-5}] \rightarrow [12_8, 12_4] \rightarrow [12_5, 12_9] \rightarrow [12_{10}, 12_6] \rightarrow \\
 &[12_7, 12_{11}] \rightarrow [12_8, 24_4] \rightarrow [24_5, 12_9] \rightarrow [12_{10}, 24_6] \rightarrow [24_7, 12_{11}] \rightarrow [6_6, 12_{12}] \rightarrow \\
 &[12_1, 6_1] \rightarrow [12_2, 6_2] \rightarrow [6_3, 12_3] \rightarrow [12_4, 6_4] \rightarrow [6_5, 12_5] \rightarrow [12_6, 6_6] \rightarrow \\
 &[6_1, 12_7] \rightarrow [6_2, (n)_{n-4}] \rightarrow [(n)_{n-3}, 6_3] \rightarrow [6_4, (n)_{n-2}] \rightarrow [(n)_{n-1}, 6_5] \rightarrow [12_{12}, (n)_n] \rightarrow \\
 &[(n)_1, 12_1].
 \end{aligned}$$

It follows that f has a periodic orbit of period 24, denoted by $24_1, \dots, 24_{24}$, and respecting the definition of the period doubling pattern, as shown in Figure 25:

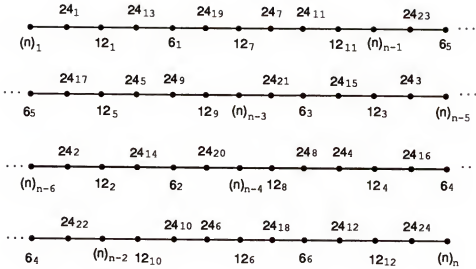


Fig. 25

Note that the intervals

$$[(n)_{n-4}, 12_8], [12_9, (n)_{n-3}], [(n)_{n-2}, 12_{10}] \text{ and } [12_{11}, (n)_{n-1}]$$

do not contain any point of period 24.

Now, using induction, assume that a period doubling pattern of the orbit of period 6 has been constructed, for orbits up to period $2^k 6$. To construct a loop of length $2^{k+1} 6$ we proceed as follows : We start with the interval $[(n)_1, (2^k 6)_1]$, and follow the orbit of $(2^k 6)_1$ up to the point $(2^k 6)_{2^k 6}$. Each time we choose the interval with endpoints $(2^k 6)_i$ ($i = 1, \dots, 2^k 6$) and the adjacent point to $(2^k 6)_i$ covered by the previous interval in the loop. Next we continue following the orbit of $(2^k 6)_1$, but now the intervals chosen are the ones lying to the opposite side of $(2^k 6)_i$, for $i = 1, \dots, 2^k 6 - 5$. Then, we follow the orbit of the point $(n)_{n-4}$ up to the point $(n)_n$. Each time we choose the interval with endpoints $(n)_j$ ($j = n - 4, \dots, n$) and the adjacent point to $(n)_j$ covered by the previous interval in the loop. Finally $[(2^k 6)_{2^k 6}, (n)_n]$ covers $[(n)_1, (2^k 6)_1]$ and the loop is complete.

Note that the intervals

$$\langle (n)_{n-4}, (2^k 6)_{2^k 6-4} \rangle, \langle (2^k 6)_{2^k 6-3}, (n)_{n-3} \rangle,$$

$$\langle (n)_{n-2}, (2^k 6)_{2^k 6-2} \rangle, \langle (2^k 6)_{2^k 6-1}, (n)_{n-1} \rangle$$

do not contain any point of period $2^{k+1}6$.

It follows that the loop obtained in this way is nonrepetitive and by construction respects the period doubling pattern centered at the periodic orbit of period 6.

This completes the induction step and the proof of the lemma. ■

With Lemma 3.3.1 and Lemma 3.4.1 in mind, the reader can easily start with any orbit of period n , where n is not a power of two, and construct a period doubling pattern centered at some orbit of period m , where m is not a power of two, and $n \prec m$ in Sarkovskii's ordering.

CHAPTER 4

EQUICONTINUITY OF ITERATES OF CIRCLE MAPS

Throughout this chapter we let X, Y denote metric spaces and $C^0(X, Y)$ denote the set of continuous maps from X to Y . Let I and S^1 denote the closed unit interval on the real line and the unit circle on the plane respectively. Finally, if $f \in C^0(X, X)$, we let F_1 and F_2 denote the fixed point set of f and f^2 respectively.

A.M. Bruckner and T. Hu [17] have investigated the case where $X = Y = I$ and have shown that the family of iterates of f i.e. $\{f, f^2, f^3, \dots\} = \{f^n\}_{n=1}^\infty$ is equicontinuous on I if and only if $F_2 = \bigcap_{n=1}^\infty f^n(I)$.

We examine the case where $X = Y = S^1$, and we show that $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if one of the following holds :

- f is conjugate to a rotation.
- F_1 consists of exactly two distinct points and every other point on S^1 has period two.
- F_1 consists of a single point and $F_2 = \bigcap_{n=1}^\infty f^n(S^1)$.
- $F_1 = \bigcap_{n=1}^\infty f^n(S^1)$.

4.1 Definitions and Results on the Interval

Let $f \in C^0(S^1, S^1)$. We think of the circle S^1 as R/Z and for $x, y \in S^1$ with $x \neq y$ we denote by $[x, y]$ the closed interval (arc) from x counterclockwise to y . Let $d(x, y)$ denote the $\min\{|[x, y]|, |[y, x]|\}$ where $|[x, y]|$ is the length of the interval $[x, y]$.

Definition 4.1.1 A family $\mathcal{F}$ of functions from a metric space X to a metric space (Y, d) is called equicontinuous at $x \in X$ if given $\epsilon > 0$, $\exists$ open set O containing x such that $d(f(x), f(y)) < \epsilon$ for all $y \in O$ and all $f \in \mathcal{F}$.

The family is said to be equicontinuous on X if it is equicontinuous at each point $x \in X$.

In our case $X = Y = S^1$ is a compact metric space. We will show that the previous definition can be formulated as follows:

Definition 4.1.2 A family $\mathcal{F}$ of functions from S^1 to itself is called equicontinuous on S^1 if given $\epsilon > 0$, $\exists \delta > 0$ such that for all $x, y \in S^1$, $d(f(x), f(y)) < \epsilon$, $\forall f \in \mathcal{F}$, whenever $d(x, y) < \delta$.

Lemma 4.1.3 For $X = Y = S^1$, Definition 4.1.1 and Definition 4.1.2 are equivalent.

Proof: Suppose that $\mathcal{F}$ is equicontinuous at each point of S^1 .

Fix $\epsilon > 0$. Then for each $z \in S^1$, $\exists \delta_z > 0$ such that $d(f(z), f(z')) < \frac{1}{2}\epsilon$, $\forall f \in \mathcal{F}$, whenever $d(z, z') < \delta_z$.

Now $\{B(z; \delta_z) : z \in S^1\}$ forms an open cover of S^1 . By Lebesgue's Covering Lemma, $\exists \delta > 0$ such that for each $x \in S^1$, $B(x; \delta)$ is contained in one of the sets of this cover. So if $x, y \in S^1$ and $d(x, y) < \delta$, $\exists z \in S^1$ with $y \in B(x; \delta) \subset B(z; \delta_z)$. It follows that $d(x, z) < \delta_z$ and $d(y, z) < \delta_z$. This gives $d(f(x), f(z)) < \frac{1}{2}\epsilon$ and $d(f(y), f(z)) < \frac{1}{2}\epsilon$ which implies that $d(f(x), f(y)) < \epsilon$ and $\mathcal{F}$ is equicontinuous on S^1 . ■

Now take $\mathcal{F}$ to be the family of iterates of f , that is $\mathcal{F} = \{f, f^2, f^3, \dots\}$. By the equivalence of the above definitions, the family of functions $\mathcal{F}$ is said to be equicontinuous if given $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in S^1$ $d(f^i(x), f^i(y)) < \epsilon$, for all $i \geq 1$, whenever $d(x, y) < \delta$.

Definition 4.1.4 Let $f \in C^0(X, X)$ and $g \in C^0(Y, Y)$. We say that the maps f and g are (topologically) conjugate, if there exists a homeomorphism $h \in C^0(X, Y)$ such that $h(f(x)) = g(h(x))$ for every $x \in X$.

The following theorem is proved by J.Cano [18]:

Theorem 4.1.5 Let $f \in C^0(I, I)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Then F_1 is connected and if it is non-degenerate then $F_1 = P(f)$.

The next theorem which is due to Bruckner and Hu [17] (only if) and Boyce [16] (if):

Theorem 4.1.6 Let $f \in C^0(I, I)$. Then $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if $\bigcap_{n=1}^\infty f^n(I) = F_2$.

Corollary 4.1.7 Let $f \in C^0(I, I)$. If f has a periodic point of period $n > 2$, then $\{f^n\}_{n=1}^\infty$ cannot be equicontinuous.

Now let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. We consider three cases :

- (I) f has a fixed point on S^1 .
- (II) the smallest period of the periodic points of f on S^1 is $n \geq 2$.
- (III) f has no periodic points on S^1 .

The following three sections consider each case separately.

4.2 The Least Existing Period is $n = 1$

We start with case (I). The basic result of this case is Theorem 4.2.5. We first show the following four lemmas:

Lemma 4.2.1 Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Suppose that there is a fixed point p on S^1 , and let J be the component of F_2 containing p . If J is

either $\{p\}$ or a proper closed interval containing p , then there exists an open interval K containing J such that $\omega(x, f) \subseteq J$, for every x in K .

Proof: First suppose that $J = \{p\}$. Let $\epsilon = |S^1|/4 > 0$.

By equicontinuity of $\{f^n\}_{n=1}^\infty$ there is an open interval K containing p such that for every x in K and for every $n \geq 1$, $d(f^n(x), p) < \epsilon$. Define $L = \overline{\bigcup_{j=0}^\infty f^j(K)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorem 4.1.5 and Theorem 4.1.6), the fixed point set of $f|_L$ and $f^2|_L$ is connected and therefore, it is $\{p\}$.

Moreover, by the above corollary all periodic points of $f|_L$ have period 1 or 2. Thus the fixed point p is the only periodic point of f in L . Therefore $P(f)|_L = F_1$ and by [28] the ω -limit points coincide with the fixed points. Hence p is the only ω -limit point of f in L and thus $\omega(x, f) = \{p\} = J$, for every x in L . Since $K \subset L$, $\omega(x, f) = \{p\} = J$, for every x in K .

Now suppose that J is a proper closed interval containing p . Let q_1 and q_2 be the endpoints of J , which are fixed points under f^2 . Let $\epsilon = |S^1 - J|/4 > 0$.

By equicontinuity of $\{f^n\}_{n=1}^\infty$, there is an open interval K_1 around q_1 such that for every x in K_1 and for every $n \geq 1$, $d(f^n(x), f^n(q_1)) < \epsilon$. Similarly there is an open interval K_2 around q_2 such that for every x in K_2 and for every $n \geq 1$, $d(f^n(x), f^n(q_2)) < \epsilon$.

Define $L = \overline{\bigcup_{j=0}^\infty f^j(K_1 \cup J \cup K_2)}$. Then L is a closed, proper, invariant interval. By previous results on the interval (Theorem 4.1.5 and Theorem 4.1.6), the fixed point set of $f|_L$ and $f^2|_L$ is connected and therefore it is J . Moreover, by Corollary 4.1.7 all periodic points of $f|_L$ have period 1 or 2, which we know that lie in J . Since $P(f)|_L$ is closed, by [28], it coincides with the set of ω -limit points.

Therefore $\omega(x, f) \subset J$, for every x in L . Let $K = K_1 \cup J \cup K_2$. Then $K \subset L$ and $\omega(x, f) \subset J$, for every x in K . ■

Lemma 4.2.2 *Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Suppose that there is a fixed point p on S^1 , and let J be the component of F_2 containing p . Define $S = \{x \in S^1 : \omega(x, f) \subseteq J\}$. Then $S = S^1$.*

Proof: There are three cases :

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$.

If (iii) holds then obviously $S = J = S^1$.

Therefore assume (i) or (ii) hold. Then by Lemma 4.2.1, there exists an open interval K containing J such that $\omega(x, f) \subseteq J$, for every x in K . Note that S is nonempty since $S \supseteq K$. First we show that S is open: Let $x \in S$. Then $\omega(x, f) \subseteq J$, by definition of S . Choose N large enough such that $f^N(x) \in K$. By continuity of f^N there is a neighborhood U of x such that if $y \in U$ then $f^N(y) \in K$. But then $\omega(f^N(y), f) = \omega(y, f) \subseteq J$ and $y \in S$. Therefore S is open.

Let T be the component of S containing J and therefore K , as well. Then T is open and connected. We will show that $T = S^1$. Suppose $T \neq S^1$. Then $S^1 - T$ is a closed interval or a point. Let $J = [q_1, q_2]$ where possibly $q_1 = q_2 = p$.

Suppose first that $S^1 - T$ is a closed interval. Let z_1 and z_2 be the endpoints of this closed interval such that $[z_2, z_1] \cap J = \emptyset$. Let $\epsilon = \frac{1}{2} \min\{d(q_1, z_1), d(q_2, z_2)\}$. By equicontinuity of $\{f^n\}$ at z_1 , there is an open interval V_1 around z_1 such that for every x in V_1 and for every $n \geq 1$, $d(f^n(x), f^n(z_1)) < \epsilon$. Let $x \in T$ such that $d(x, z_1) < \epsilon$. Since $\omega(x, f) \subseteq J$ and the orbit of z_1 stays by definition out of T , there exists a positive integer k such that $d(f^k(x), f^k(z_1)) > \epsilon$, which is a contradiction.

Now suppose that $S^1 - T = \{z\}$. Let $\epsilon = \frac{1}{2} \min\{d(z, q_1), d(q_2, z)\}$. By equicontinuity of $\{f^n\}$ at z there is an open interval V around z such that for every x in V and for every $n \geq 1$, $d(f^n(x), f^n(z)) < \epsilon$. Since $\omega(x, f) \subseteq J$ for every $x \in T$ and $f(z) = z$ is a fixed point of f , we get a contradiction.

Hence $T = S^1$. Thus $S = \{x \in S^1 : \omega(x, f) \subseteq J\} = S^1$. ■

Lemma 4.2.3 *Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$, then F_1 cannot consist of exactly one point.*

Proof: Suppose that there is an $f \in C^0(S^1, S^1)$ such that $F_2 = S^1$ and $F_1 = \{p\}$. Let z be a point on $S^1 - \{p\}$ of period two. Let K be the closed interval with endpoints z and $f(z)$ which contains p and let L be the closed interval with the same endpoints that does not contain p . Since f is a homeomorphism, we have two cases:

(i) $f(K) = K$ and $f(L) = L$ or (ii) $f(K) = L$ and $f(L) = K$.

If (i) holds then, since $f(L) = L$, there would be another fixed point of f in L , which is a contradiction since $F_1 \subset K$.

If (ii) holds then $f(K) = L$ implies that p cannot be a fixed point which is again a contradiction. ■

Lemma 4.2.4 *Let $f \in C^0(S^1, S^1)$. If $F_2 = S^1$ and F_1 consists of more than two distinct points, then f is the identity on S^1 .*

Proof: If F_1 is a proper closed set of S^1 , then $S^1 - F_1$ is an open set in S^1 . Let (a, b) denote a component of $S^1 - F_1$. Then (a, b) is free of fixed points.

Note that (a, b) has to be f -invariant, otherwise there would be at least one point of (a, b) mapping onto a fixed point, which is a contradiction to the hypothesis that $F_2 = S^1$. On the other hand, if $f(a, b) \subset (a, b)$, then since $F_2 = S^1$, (a, b) cannot be free of fixed points. Hence, we must have $F_1 = S^1$, and f is the identity on S^1 . ■

Theorem 4.2.5 *Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Suppose that there is a fixed point p on S^1 . Then f has periodic points of period at most two and F_2 is connected. Furthermore, either F_1 is connected or it consists of exactly two distinct points and every other point on S^1 has period two. Moreover, if F_1 is a nondegenerate interval then $F_1 = P(f)$.*

Proof: Let J be the component of F_2 containing p . There are three cases :

(i) $J = \{p\}$, (ii) J is a proper closed interval containing p and (iii) $J = S^1$

Assume that (i) holds. Then, by Lemma 4.2.2, $\omega(x, f) = \{p\}$ for every $x \in S^1$.

Thus the fixed point p is the only periodic point of f on S^1 , and hence $P(f) = F_1 = F_2 = \{p\}$ is connected.

Assume that (ii) holds. Then, by Lemma 4.2.2, $\omega(x, f) \subseteq J$ for every $x \in S^1$ and the periodic points of f on S^1 lie in J . By results on the interval applied to $f|_J$, either p is the unique fixed point of f on S^1 or F_1 is a nondegenerate interval and $P(f) = F_1$ on S^1 . In particular, both F_1 and F_2 are connected.

Assume that (iii) holds. Then all of the points of S^1 are periodic with period 1 or 2 and F_2 is connected. By Lemma 4.2.3, F_1 cannot consist of one point and by Lemma 4.2.4 if F_1 consists of more than two points then f is the identity map. Otherwise F_1 consists of exactly two distinct points and every other point on S^1 has period two. ■

4.3 The Least Existing Period is $n > 1$

We now investigate case (II) where the smallest period of the periodic points of f on S^1 is $n \geq 2$. The main result here is Theorem 4.3.1. We use Lemma 4.3.2 in the proof of Theorem 4.5.4.

Theorem 4.3.1 *Let $f \in C^0(S^1, S^1)$ such that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Suppose that the smallest period of the periodic points of f on S^1 is $n \geq 2$. Then every point on S^1 is periodic with period n .*

Proof: Let p be a periodic point of period n on S^1 . Then $f^n(p) = p$ and therefore p is a fixed point of f^n . Applying Theorem 4.2.5 to f^n , we conclude that F_n is either connected or it consists of exactly two distinct points and every other point on S^1 has period $2n$.

We will show that the second case does not occur. We claim that there is no continuous map of the circle having two points of period two and every other point periodic of period four. Otherwise, if g is such a map, let $p, g(p)$ be the two points of period two and let $K = [p, g(p)]$ and $L = [g(p), p]$. Since g is a homeomorphism, we have two cases: (i) $g(K) = K$ and $g(L) = L$ or (ii) $g(K) = L$ and $g(L) = K$. In both cases $g^2(K) = K$. Hence if $x \in K$ is a point of period four, then $g^2(x) \neq x$ and $g^2(x) \in K$. If M is the closed interval with endpoints x and $g^2(x)$ lying in K then $g^2(M) = M$. Therefore M contains a periodic point of period two, contradicting the assumption that p and $g(p)$ are the only points of period two and every other point has period four.

Hence F_n is connected. Suppose that $F_n \neq S^1$. Then F_n is a proper closed interval containing the orbit of p under f . Moreover $f(F_n) \subset F_n$. This implies that f has a fixed point on S^1 , contradicting the hypothesis that the smallest possible period of the periodic points is $n > 1$. Hence $F_n = S^1$. ■

For a proof of the following see [26](p 32-38).

Lemma 4.3.2 *Let $f \in C^0(S^1, S^1)$. Suppose that there exists a positive integer $n \geq 2$ such that every point on S^1 is periodic with period n . Then f is conjugate to a rational rotation.*

4.4 Lack of Periodic Points

Now we consider case (III) where $f \in C^0(S^1, S^1)$ has no periodic points and $\{f^n\}$ is equicontinuous. The main result here is listed in Theorem 4.4.2.

Note that f must be onto, since otherwise $f(S^1) = I$ is homeomorphic to a closed interval and $f(I) \subset I$, so f has a fixed point. We shall adapt the techniques and use results due to J. Auslander and Y. Katznelson [3].

Let $x \in S^1$. Then, J_x is defined to be the largest interval containing x such that $f^m(x) \notin J_x$, $\forall m \geq 1$. Denote by z_1 and z_2 the endpoints of J_x , where possibly $z_1 = z_2 = x$. The following properties are valid [3]:

J_x is closed and $z_1, z_2 \neq f^k(x)$ for $k \geq 1$.

If $x, y \in S^1$, then $y \in \omega(x, f)$ if and only if y is an endpoint of J_y .

If z_1 and z_2 are the endpoints of J_x , then $f(z_1)$ and $f(z_2)$ are the endpoints of $f(J_x)$.

$f(J_x) \cap J_x = \emptyset$ and $f^m(J_x) = J_{f^m(x)}$, $\forall m \geq 1$.

The intervals $J_{f^m(x)}$ ($m = 0, 1, 2, \dots$) are pairwise disjoint and if $f(x) = f(x')$, then $J_x = J_{x'}$.

The sets $\{J_x\}$ form a partition of S^1 (that is, if $x, y \in S^1$ then $J_x = J_y$ or $J_x \cap J_y = \emptyset$ and $\cup_{x \in S^1} J_x = S^1$).

Finally, at most countably many of the sets J_x are non-degenerate ($J_x \neq \{x\}$).

Before we show our result, we state the following theorem proved in [24] which concerns homeomorphisms.

Theorem 4.4.1 *Let f be an orientation preserving homeomorphism of S^1 to itself. For $x \in S^1$, let $R_\alpha(x) = x + \alpha \pmod{1}$ denote irrational rotation by α . Then f is conjugate to some R_α if and only if some (all) orbits of f are dense on S^1 .*

We are now ready to show the following:

Theorem 4.4.2 *Suppose $f \in C^0(S^1, S^1)$ has no periodic points, and $\{f^n\}_{n=1}^\infty$ is equicontinuous. Then f is conjugate to an irrational rotation R_α .*

Proof: We first show that $\omega(x, f) = S^1$ for all $x \in S^1$. Since $y \in \omega(x, f)$ if and only if y is an endpoint of J_y , it suffices to show that $\forall y \in S^1$, $J_y = \{y\}$. By the way of contradiction assume that J_{y_0} is a non-degenerate interval. Since f is onto, there exists $y_1 \in S^1$ such that $f(J_{y_1}) = J_{y_0}$. Continuing in this way, we obtain a sequence of intervals $\{J_{y_n}\}$ such that $f(J_{y_n}) = J_{y_{n-1}}$ $\forall n \geq 1$. Since these intervals are pairwise

disjoint, $\lim_{k \rightarrow \infty} |J_{y_k}| = 0$. Hence f^k maps arbitrarily small intervals onto J_{y_0} (as $k \rightarrow \infty$) which contradicts equicontinuity.

Therefore $\omega(x, f) = S^1$ for all $x \in S^1$. This is equivalent to saying that all orbits of f are dense in S^1 . If $f(y) = f(y')$, then $J_y = J_{y'}$ and hence f is a homeomorphism. By Theorem 4.4.1 it follows immediately that f is conjugate to an irrational rotation R_α . ■

4.5 Proof of Theorem

We first state and prove the following three lemmas which can be shown that hold on any compact metric space.

Lemma 4.5.1 *Let $f, g \in C^0(X, X)$, where X is a compact metric space. Suppose that f is conjugate to g and that $\{g^n\}_{n=1}^\infty$ is equicontinuous. Then $\{f^n\}_{n=1}^\infty$ is equicontinuous.*

Proof: Since f is conjugate to g , there exists a continuous map $h : X \rightarrow X$ such that $h^{-1} : X \rightarrow X$ is continuous and $f = h^{-1}gh$. Note that h and h^{-1} are continuous on a compact space, so that they are uniformly continuous. Let $\epsilon > 0$.

By uniform continuity of h^{-1} , there exists $\delta_1 > 0$ such that, if $d(x, y) < \delta_1$, then $d(h^{-1}(x), h^{-1}(y)) < \epsilon$ for every x, y in X .

By equicontinuity of $\{g^n\}$, there exists $\delta_2 > 0$ such that, if $d(x, y) < \delta_2$, then $d(g^n(x), g^n(y)) < \delta_1$ for every x, y in X and for every $n \geq 1$.

By uniform continuity of h , there exists $\delta_3 > 0$ such that, if $d(x, y) < \delta_3$, then $d(h(x), h(y)) < \delta_2$ for every x, y in X .

Now choose $\delta = \delta_3$. It follows that if $d(x, y) < \delta$, $d(h^{-1}g^n h(x), h^{-1}g^n h(y)) < \epsilon$ for every x, y in X and every $n \geq 1$. Since $f^n = h^{-1}g^n h$, the family $\{f^n\}_{n=1}^\infty$ is equicontinuous. ■

Lemma 4.5.2 *Let $f \in C^0(X, X)$, where X is a compact metric space. Let k be a positive integer and $g = f^k$. Then $\{f^n\}_{n=1}^\infty$ is equicontinuous if and only if $\{g^n\}_{n=1}^\infty$ is equicontinuous.*

Proof: Suppose that $\{f^n\}_{n=1}^\infty$ is equicontinuous. Then given $\epsilon > 0$ there exists $\delta > 0$ such that for any x, y with $d(x, y) < \delta$, $d(f^n(x), f^n(y)) < \epsilon$ for every $n \geq 1$. In particular $d(f^{kn}(x), f^{kn}(y)) < \epsilon$ for every $n \geq 1$ and thus $\{g^n\}_{n=1}^\infty$ is equicontinuous.

Conversely, suppose that $\{g^n\}_{n=1}^\infty = \{f^{kn}\}_{n=1}^\infty$ is equicontinuous.

Let $\epsilon > 0$. By equicontinuity of $\{f^{kn}\}_{n=1}^\infty$ there exists $\delta > 0$ such that for any x, y in X with $d(x, y) < \delta$, we have that $d(f^{kn}(x), f^{kn}(y)) < \epsilon$, $\forall n \geq 1$.

By uniform continuity of f there exists $0 < \delta_1 < \delta$ such that for any x_1, y_1 with $d(x_1, y_1) < \delta_1$, $d(f(x_1), f(y_1)) < \delta$.

By uniform continuity of f^2 there exists $0 < \delta_2 < \delta$ such that for any x_2, y_2 with $d(x_2, y_2) < \delta_2$, $d(f^2(x_2), f^2(y_2)) < \delta$.

In general, by uniform continuity of f^i there exist $0 < \delta_i < \delta$ such that for any x_i, y_i with $d(x_i, y_i) < \delta_i$, $d(f^i(x_i), f^i(y_i)) < \delta$, for $i = 1, 2, \dots, k-1$.

Let x, y be any two points in X with the property that $d(x, y) < \gamma = \min\{\delta_1, \delta_2, \dots, \delta_{k-1}\}$. We want to show that $d(f^m(x), f^m(y)) < \epsilon$ for every $m \geq 1$. If m is a multiple of k we are done by equicontinuity of $\{f^{kn}\}_{n=1}^\infty$. Otherwise, m can be written as $m = nk + i$ where n is a nonnegative integer and i is an integer with $1 \leq i \leq k-1$. Then $d(f^i(x), f^i(y)) < \delta_i$ and $d(f^{nk}(f^i(x)), f^{nk}(f^i(y))) < \epsilon$, i.e. $d(f^m(x), f^m(y)) = d(f^{nk+i}(x), f^{nk+i}(y)) < \epsilon$. Hence $\{f^m\}_{m=1}^\infty$ is equicontinuous. ■

Lemma 4.5.3 *Let $f \in C^0(X, X)$, where X is a compact metric space. If $\{(f|_{f(X)})^n\}_{n=1}^\infty$ is equicontinuous, then $\{f^n\}_{n=1}^\infty$ is equicontinuous.*

Proof: If $f(X) = X$ there is nothing to prove. Suppose that $f(X) \subset X$. Let $\epsilon > 0$.

By equicontinuity of $\{(f|_{f(X)})^n\}$ there exists $0 < \delta < \epsilon$ such that for any x, y in $f(X)$ with $d(x, y) < \delta$, we have that $d((f|_{f(X)})^n(x), (f|_{f(X)})^n(y)) < \epsilon$, $\forall n \geq 1$.

By uniform continuity of f there exists $0 < \delta_1 < \delta$ such that for any x_1, y_1 in X with $d(x_1, y_1) < \delta_1$, $d(f(x_1), f(y_1)) < \delta$.

Let x, y be any two points in X with $d(x, y) < \delta_1$. Then $d(f(x), f(y)) < \delta < \epsilon$ and $f(x), f(y) \in f(X)$. Moreover $d(f^n(f(x)), f^n(f(y))) = d(f^{n+1}(x), f^{n+1}(y)) < \epsilon$, $\forall n \geq 1$.

In particular, $d(f^n(x), f^n(y)) < \epsilon$, $\forall n \geq 1$ and thus $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

■

Finally, we summarize the results in the following theorem.

Theorem 4.5.4 *Let $f \in C^0(S^1, S^1)$. Then $\{f^n\}_{n=1}^{\infty}$ is equicontinuous if and only if one of the following holds:*

- (1) f is conjugate to a rotation.
- (2) F_1 consists of exactly two distinct points and every other point on S^1 has period two.
- (3) F_1 consists of a single point and $F_2 = \bigcap_{n=1}^{\infty} f^n(S^1)$.
- (4) $F_1 = \bigcap_{n=1}^{\infty} f^n(S^1)$.

Proof: We suppose that $\{f^n\}$ is equicontinuous. First assume that $F_1 = \emptyset$.

If f has no periodic points on S^1 , then by Theorem 4.4.2, f is conjugate to an irrational rotation, so that (1) holds.

If the smallest period of the periodic points of f on S^1 is $n \geq 2$, then by Theorem 4.3.1, every point on S^1 is periodic with period n . It follows by Lemma 4.3.2, that f is conjugate to a rational rotation, so that (1) holds again.

Now assume that $F_1 \neq \emptyset$. Then by Theorem 4.2.5, f has periodic points of period at most two. If F_1 is not connected, then by Theorem 4.2.5, F_1 consists of exactly two distinct points, and every other point of S^1 has period two, so that (2) holds.

If F_1 is connected, then it consists of (i) a single point, (ii) a proper interval or (iii) the whole circle.

(i) First assume that F_1 consists of a single point p . Notice that by Theorem 4.2.5 and Lemma 4.2.3, F_2 is a connected proper interval of S^1 .

Moreover by Lemma 4.2.2, we have that for every $x \in S^1$, $\omega(x, f) \subseteq F_2$. As in the proof of Lemma 4.2.1, there exists an open interval K containing F_2 , such that, if $L = \overline{\bigcup_{j=0}^{\infty} f^j(K)}$, then L is a proper interval. Of course, L is also closed and invariant.

For $x \in S^1$, since $\omega(x, f) \subseteq F_2 \subset K \subset L$, there exists a positive integer N such that $f^N(x) \in K$. Then $f^m(x) \in L$ for every $m \geq N$.

By continuity of f^N , there exists an open neighborhood V_x of x , such that $f^N(V_x) \in K$, and hence $f^m(V_x) \in L$, for every $m \geq N$. Note that for each $x \in S^1$, the collection $\{V_x\}_{x \in S^1}$ forms an open cover of S^1 .

By compactness of S^1 , there exists a finite subcover, which we denote by $\{V_i\}_{i=1, \dots, l}$. Consequently, for every V_i there exists a positive integer N_i , such that $f^{N_i}(V_i) \subset K$, for $i = 1, 2, \dots, l$ and $f^{m_i}(V_i) \subset L$, for every $m_i \geq N_i$ and for $i = 1, 2, \dots, l$. Choose $N = \max\{N_1, \dots, N_l\}$. Then $f^m(V_i) \subset L$, for every $m \geq N$ and $i = 1, 2, \dots, l$. Thus $f^m(S^1) \subset L$ for every $m \geq N$.

By Theorem 4.1.6, $\bigcap_{n=1}^{\infty} f^n(L) = F_2$. Since $f^m(S^1) \subset L$, for every $m \geq N$, it follows that $\bigcap_{n=1}^{\infty} f^n(L) = \bigcap_{n=1}^{\infty} f^n(S^1) = F_2$. Hence (3) holds.

(ii) Now assume that F_1 is a proper interval of S^1 . We know by Theorem 4.2.5, that F_1 coincides with the set of periodic points of f . By an argument similar to the above applied to F_1 , we can see that $F_1 = \bigcap_{n=1}^{\infty} f^n(S^1)$, and hence (4) holds.

(iii) If $F_1 = S^1$, then obviously (4) holds again.

This concludes one direction of the proof, namely that if $\{f^n\}_{n=1}^{\infty}$ is equicontinuous then one of (1), (2), (3) or (4) holds. Now we will show that all of these four cases, imply that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Suppose that (1) holds, i.e. f is conjugate to a rotation R . Then R^n is an isometry for every $n \geq 1$, and therefore $\{R^n\}$ is equicontinuous. It follows by Lemma 4.5.1, that $\{f^n\}$ is equicontinuous as well.

Suppose that (2) holds, i.e. F_1 consists of exactly two distinct points, and every other point on S^1 has period two. Then f^2 is the identity on S^1 . Therefore $\{f^{2n}\}$ is equicontinuous. It follows by Lemma 4.5.2, that $\{f^n\}$ is equicontinuous as well.

Suppose that (3) holds, i.e. F_1 consists of a single point and $F_2 = \cap_{n=1}^{\infty} f^n(S^1)$. Then $f(S^1) \neq S^1$, since otherwise $F_2 = S^1$, and we have seen in Lemma 4.2.3, that there is no continuous map of the circle with one fixed point and every other point of period two. Hence $f(S^1)$ is a proper interval of S^1 , and $f|_{f(S^1)} : f(S^1) \rightarrow f(S^1)$ is a continuous map of the interval, with fixed point set of $(f|_{f(S^1)})^2$ equal to F_2 . Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n(f(S^1)) = F_2$, it follows by Theorem 4.1.6, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 4.5.3, we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous.

Finally suppose that (4) holds, i.e. $F_1 = \cap_{n=1}^{\infty} f^n(S^1)$. If $S^1 = f(S^1)$, then $F_1 = S^1$, and the identity map is equicontinuous. If $f(S^1)$ is a proper interval of S^1 , then F_1 is a point or a proper interval of S^1 . It follows that $f|_{f(S^1)} : f(S^1) \rightarrow f(S^1)$ is a continuous map of the interval, such that its fixed point set equals the fixed point set of f on S^1 .

Since $\cap_{n=1}^{\infty} (f|_{f(S^1)})^n(f(S^1)) = F_1$, it follows by Theorem 4.1.6, that $\{(f|_{f(S^1)})^n\}_{n=1}^{\infty}$ is equicontinuous. By Lemma 4.5.3, we get that $\{f^n\}_{n=1}^{\infty}$ is equicontinuous. ■

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BIOGRAPHICAL SKETCH

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